Near-inertial waves in strongly baroclinic currents: theory and oceanic applications

Dan Whitt

dw494@cam.ac.uk

UNIVERSITY OF CAMBRIDGE

With support from:

NSF international
post-doctoral fellowship to D Whitt
Grants to L Thomas

Grants to L Thomas to
collect and analyze data

PhD tuition and stipend support to D Whitt
from William Whiteford Fellowship Fund
Stanford University

Near-Inertial Waves in Strongly Baroclinic Currents

DANIEL B. WHITT AND LEIF N. THOMAS

Environmental Earth System Science Department, Stanford University, Stanford, California

(Manuscript received 24 July 2012, in final form 12 December 2012)

ABSTRACT

An analysis and physical interpretation of near-inertial waves (NIWs) propagating perpendicular to a steady, two-dimensional, strongly baroclinic, geostrophic current are presented. The analysis is appropriate for geostrophic currents with order-one Richardson numbers such as those associated with fronts experiencing strong, wintertime atmospheric forcing. This work highlights the underlying physics behind the properties of the NIWs using parcel arguments and the principles of conservation of density and absolute momentum. Baroclinicity introduces lateral gradients in density and vertical gradients in absolute momentum that significantly modify the dispersion and polarization relations and propagation of NIWs relative to classical internal wave theory. In particular, oscillations at the minimum frequency are not horizontal but, instead, are slanted along isopycnals. Furthermore, the polarization of the horizontal velocity is not necessarily circular at the minimum frequency and the spiraling of the wave's velocity vector with time and depth can be in the opposite direction from that predicted by classical theory. Ray tracing and numerical solutions illustrate the trapping and amplification of NIWs in regions of strong baroclinicity where the wave frequency is lower than the effective Coriolis frequency. The largest amplification is found at slantwise critical layers that align with the tilted isopycnals of the current. Such slantwise critical layers are seen in wintertime observations of the Gulf Stream and, consistent with the theory, coincide with regions of intensified ageostrophic shear characterized by a banded structure that is spatially coherent along isopycnals.

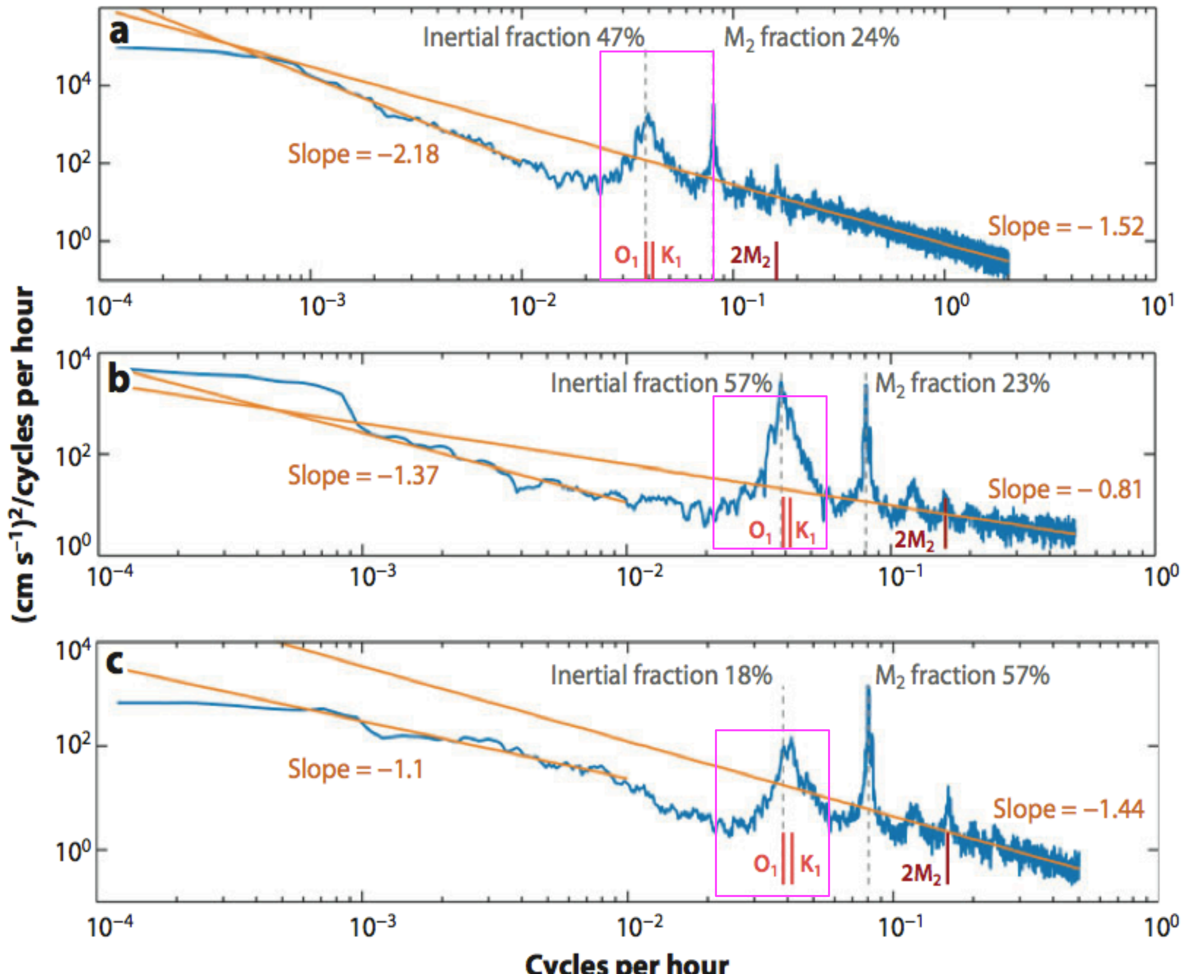
Oceanic KE frequency spectra peaked at

$$\omega \approx f = 2\Omega_{earth} \sin(latitude)$$

128 m

1500 m

3900 m



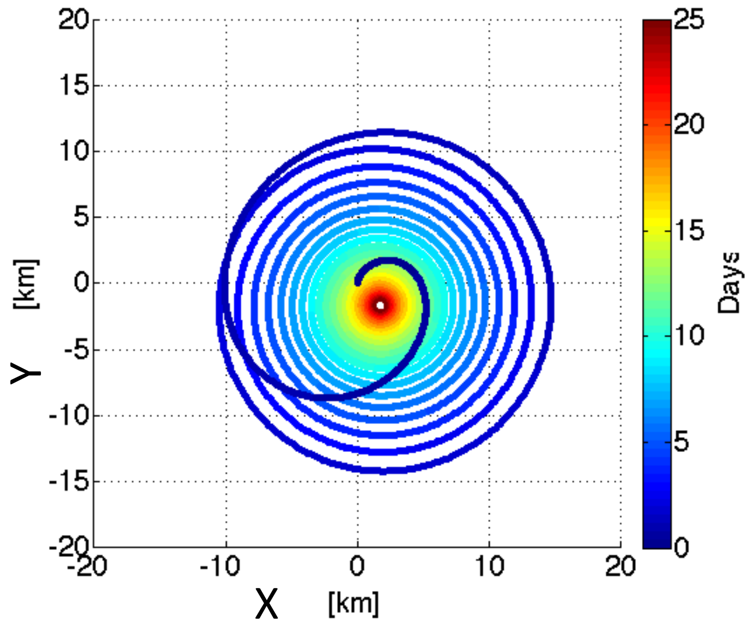
- **5-10% of all KE**
- Surface intensified, but present at all depths
- "Despite their ubiquity, energy, and many years of study, much about the behavior of inertial waves remains obscure." **[Ferrari and Wunsch 2009]**

Surface currents resonate at

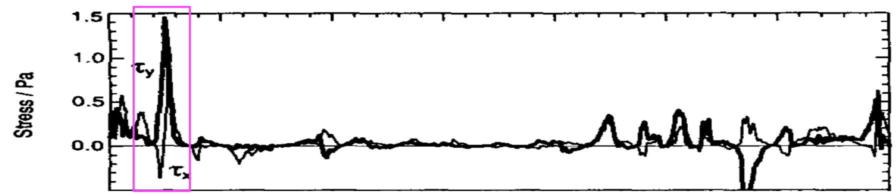
$$f = 2\Omega_{earth}\sin(latitude)$$

Slab-Model Surface Drifter Trajectory

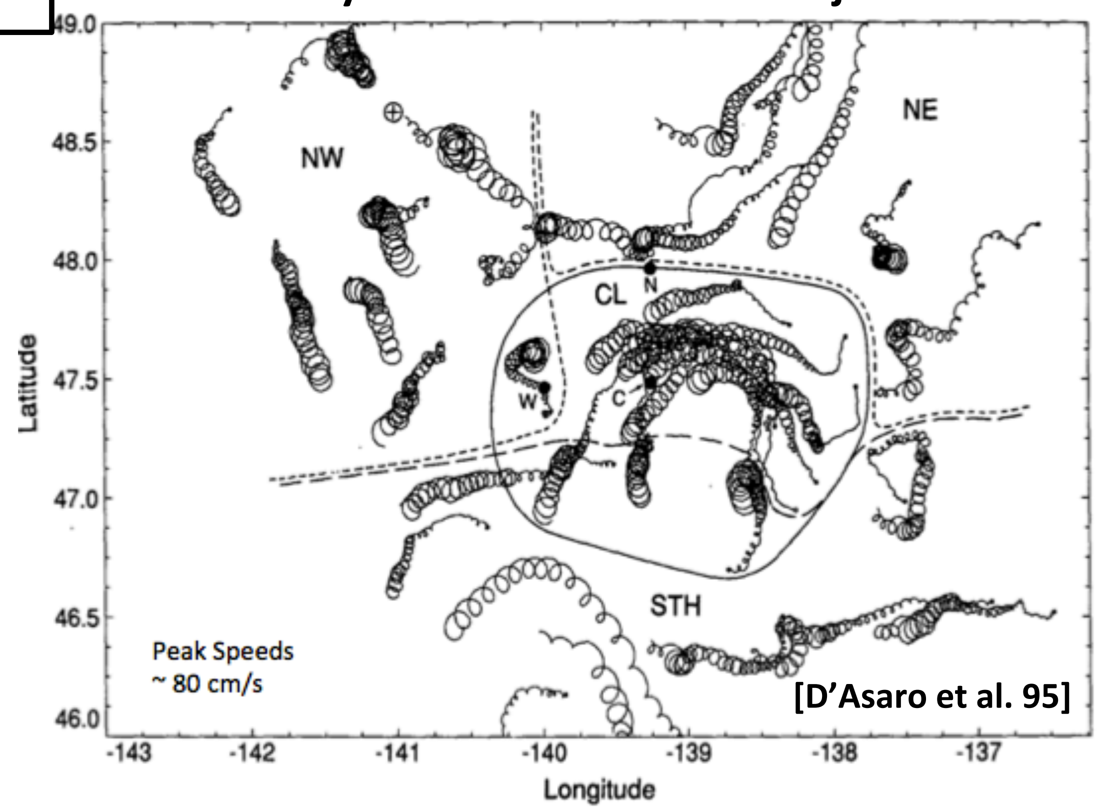
$$\boxed{\frac{\partial \mathbf{U}_{ML}}{\partial t} + f\mathbf{k}\times\mathbf{U}_{ML} = \frac{\boldsymbol{\tau}}{\rho H_{ML}} - r\mathbf{U}_{ML}}$$



35 days of observed surface wind stress



25 days of observed drifter trajectories



[D'Asaro et al. 95]

Mixed layer near-inertial currents are amplified under atmospheric storm tracks

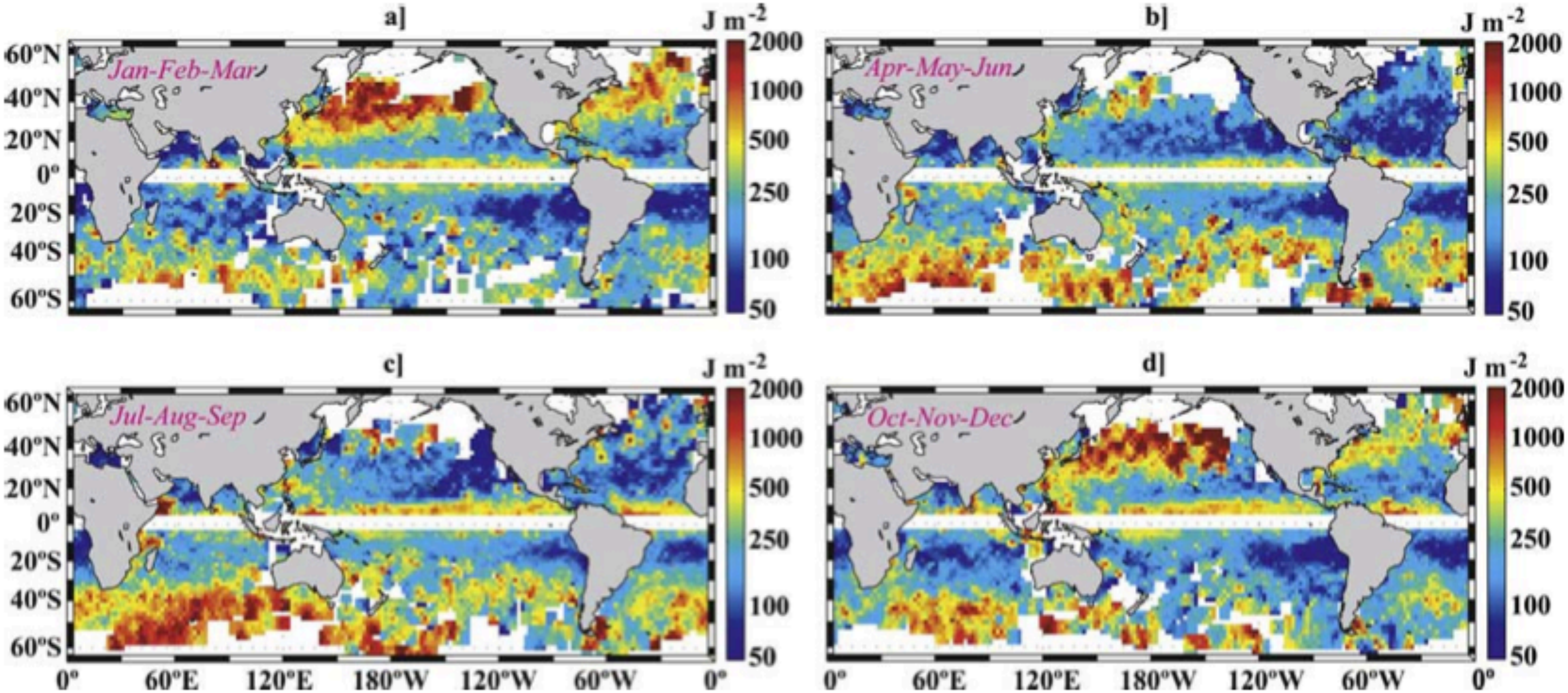


Figure 2. Seasonal variation of inertial mixed-layer energy computed from satellite-tracked drifter trajectories.

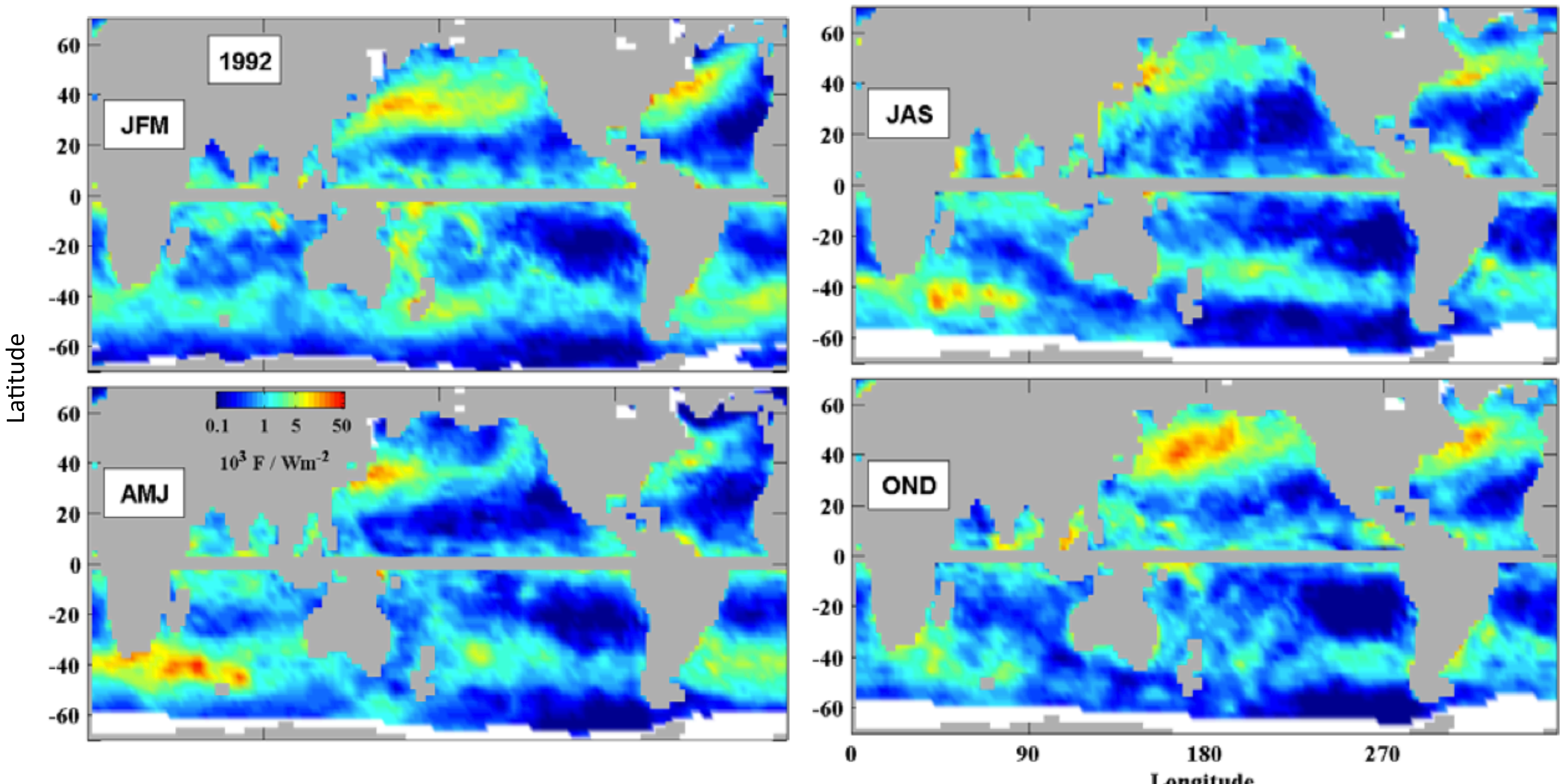
3 of 5

Especially during winter

[Chaigneau et al. 2008]

KE flux from winds to mixed layer inertial currents qualitatively consistent with drifter observations

[Alford 2003]

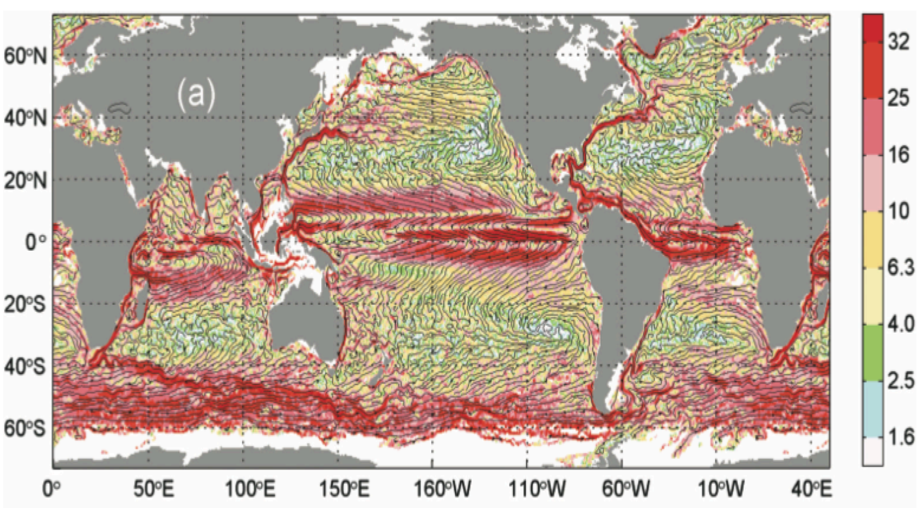


Calculated using 6-hr NCEP reanalysis winds
Input to the slab model

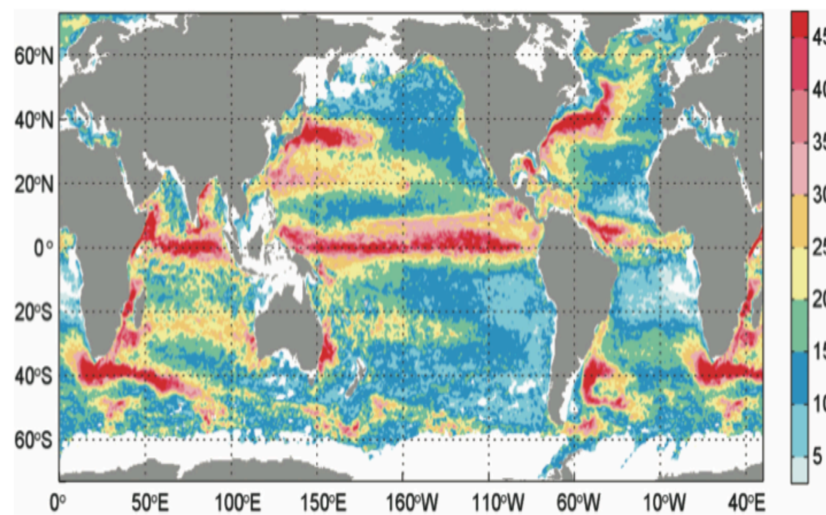
$$\frac{\partial \mathbf{U}_{ML}}{\partial t} + f\mathbf{k} \times \mathbf{U}_{ML} = \frac{\tau}{\rho H_{ML}} - r\mathbf{U}_{ML}$$

Near-inertial motions coexist with energetic lower-frequency geostrophic flows

Mean surface currents [cm/s]



St. dev. surface currents [cm/s]



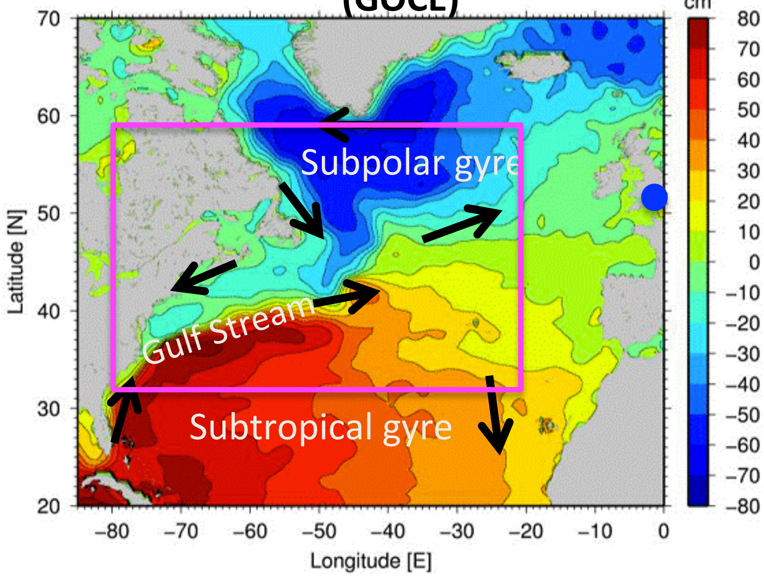
~90% of KE in ocean: **balanced** low-frequency mesoscale eddies and mean flows
~10 % of KE: **ageostrophic** near-inertial motions

Big questions:

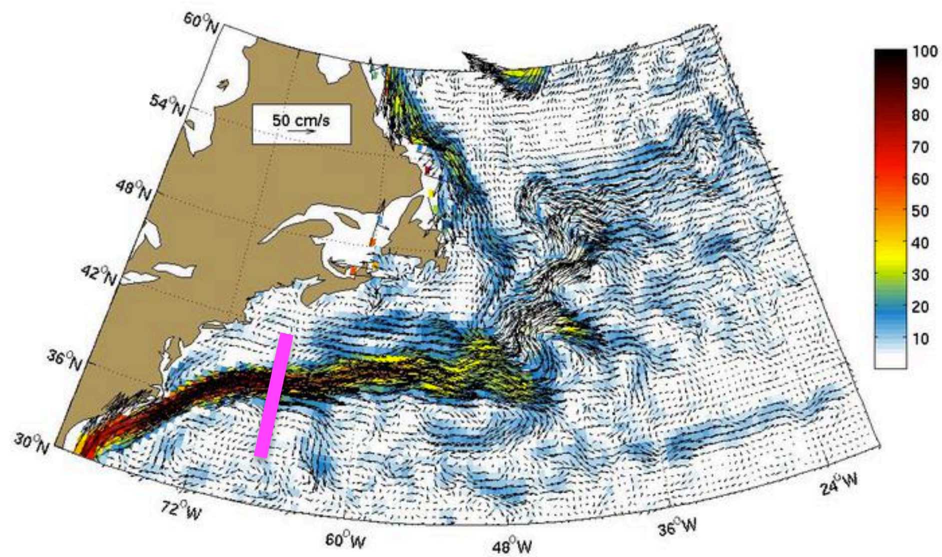
1) Kinetic Energy – do balanced flows provide a significant source of KE for NIW?
2) Upper-ocean mixing – do balanced flows modulate wind-generated NIW and boundary-layer turbulence, ocean heat, nutrient, tracer budgets, atmosphere-ocean exchange?

For Example: the Gulf Stream

Mean Dynamic Sea Surface Height anomaly h (GOCE)



Mean Surface Current Speed |u| (drifters)



http://oceancurrents.rsmas.miami.edu/atlantic/

- Sharp drop in sea surface height (~ 1 m)
- Strong mean current (~ 1 m/s)

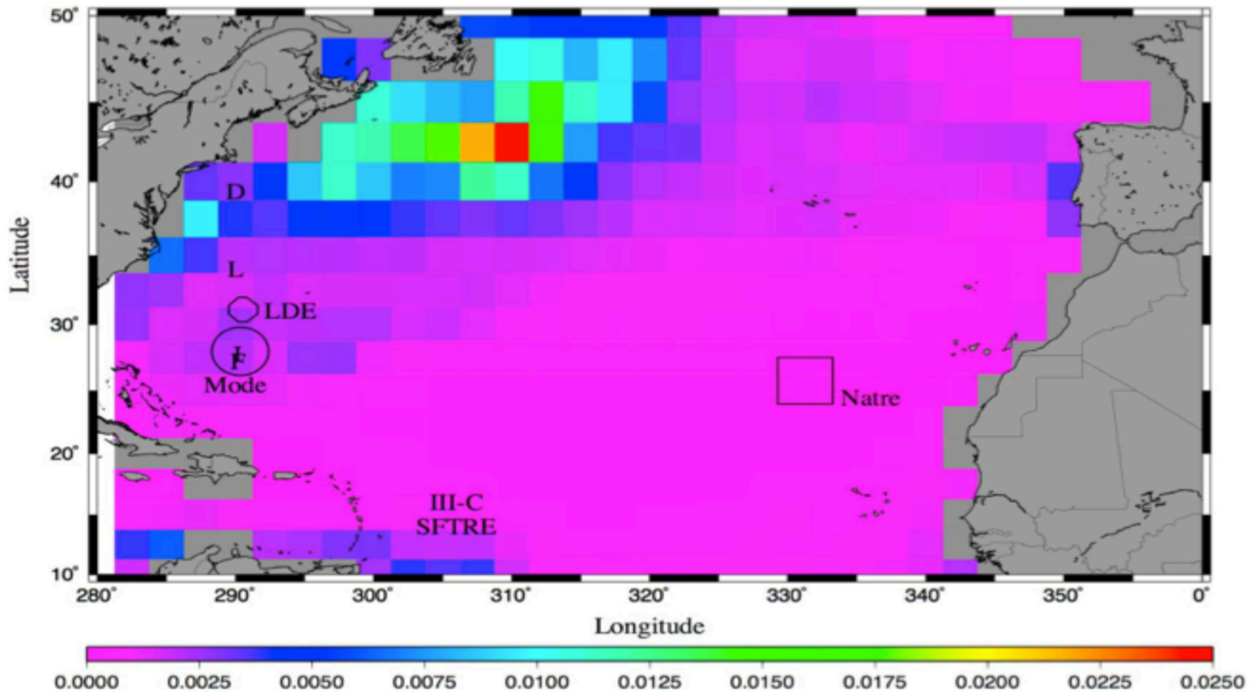
Approximate geostrophic force balance just below the surface boundary layer

Coriolis Force   Pressure gradient force

$$\mathbf{f} \times \langle \mathbf{u} \rangle = -g \nabla \langle h \rangle$$

Annual average KE from winds to NIW in North Atlantic

(NCEP/NCAR reanalysis) **[e.g. Alford 2003]**



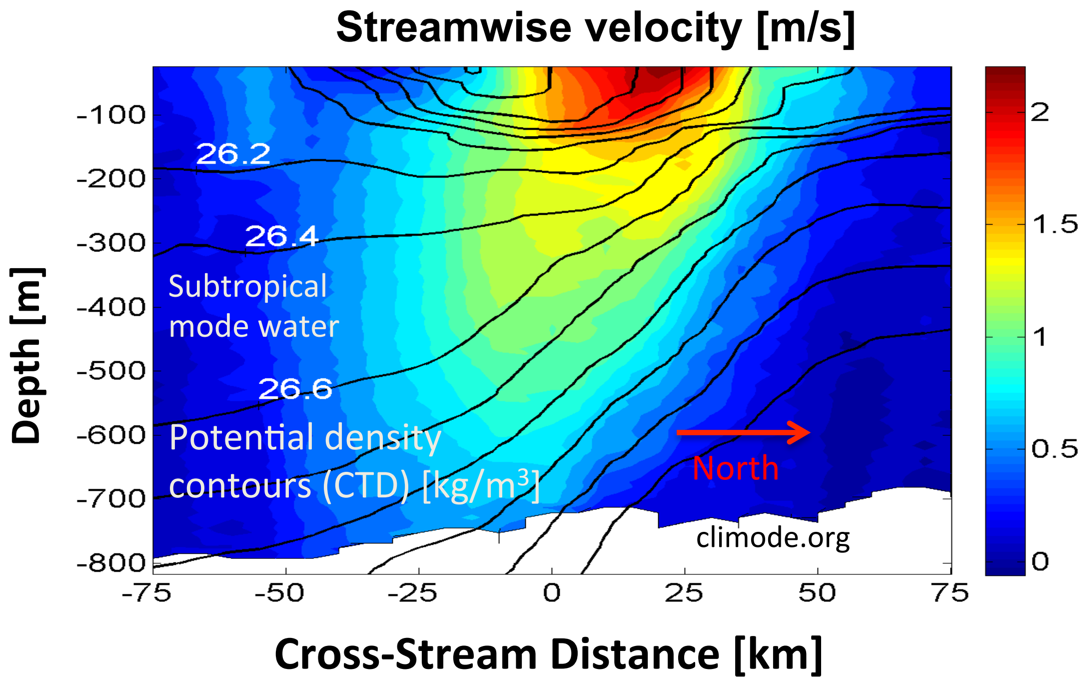
0 [W/m^2] .025

- Gulf Stream lies underneath atmospheric storm tracks.

A strongly baroclinic geostrophic jet

Momentum Eqns

$$\frac{D\mathbf{u}}{Dt} + \boxed{\mathbf{f}\times\mathbf{u} = -\frac{\nabla p}{\rho_0} + \mathbf{k}b} + \Im$$



Density is written an anomaly from 1000 kg/m^3

Buoyancy: $b = -\frac{g\rho}{\rho_0}$

- Surface pressure gradient compensated by baroclinic pressure gradient at depth.
- Velocity sheared, nearly in **thermal wind balance**.

Geostrophic balance: $fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$

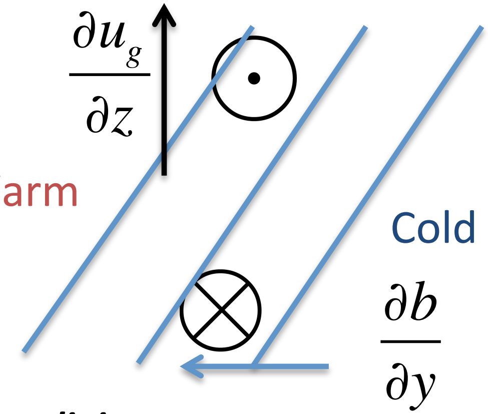
Hydrostatic Balance: $0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + b$

Thermal Wind Balance

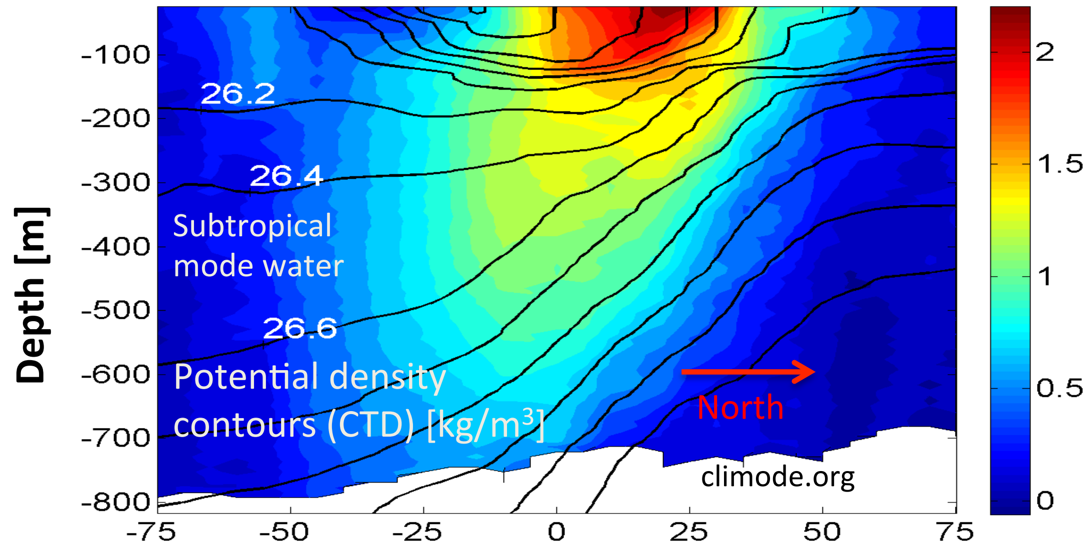
$$f\frac{\partial u}{\partial z} = -\frac{\partial b}{\partial y}$$

A strongly baroclinic geostrophic jet

Physics of Thermal Wind Balance

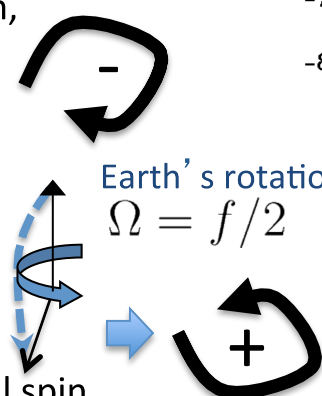


Streamwise velocity [m/s]



Cross-Stream Distance [km]

Baroclinic torque
Without planetary rotation, density contours slump over



Tilting of planetary vorticity
Vertically sheared geostrophic flow tilts vertical spin into horizontal spin balancing baroclinic torque

Density is written an anomaly from 1000 kg/m³

Buoyancy: $$b = -\frac{g\rho}{\rho_0}$$

- Surface pressure gradient compensated by baroclinic pressure gradient at depth.
- Velocity sheared, nearly in **thermal wind balance**.

Geostrophic balance: $$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Hydrostatic Balance: $$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + b$$

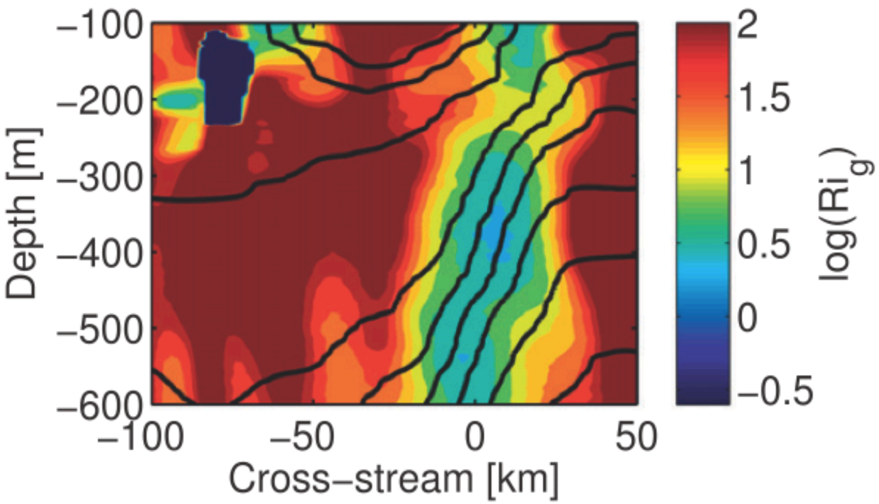
Thermal Wind Balance

$$f\frac{\partial u}{\partial z} = -\frac{\partial b}{\partial y}$$

A strongly baroclinic geostrophic jet

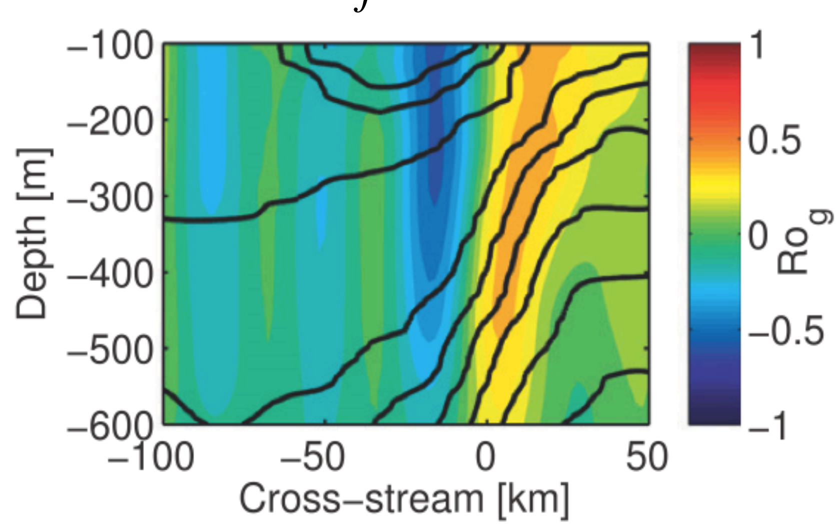
Geostrophic Richardson number

$$Ri_g = N^2 \Big/ \left|\partial \mathbf{u}_g / \partial z\right|^2 \sim 1-10$$



Geostrophic Rossby number

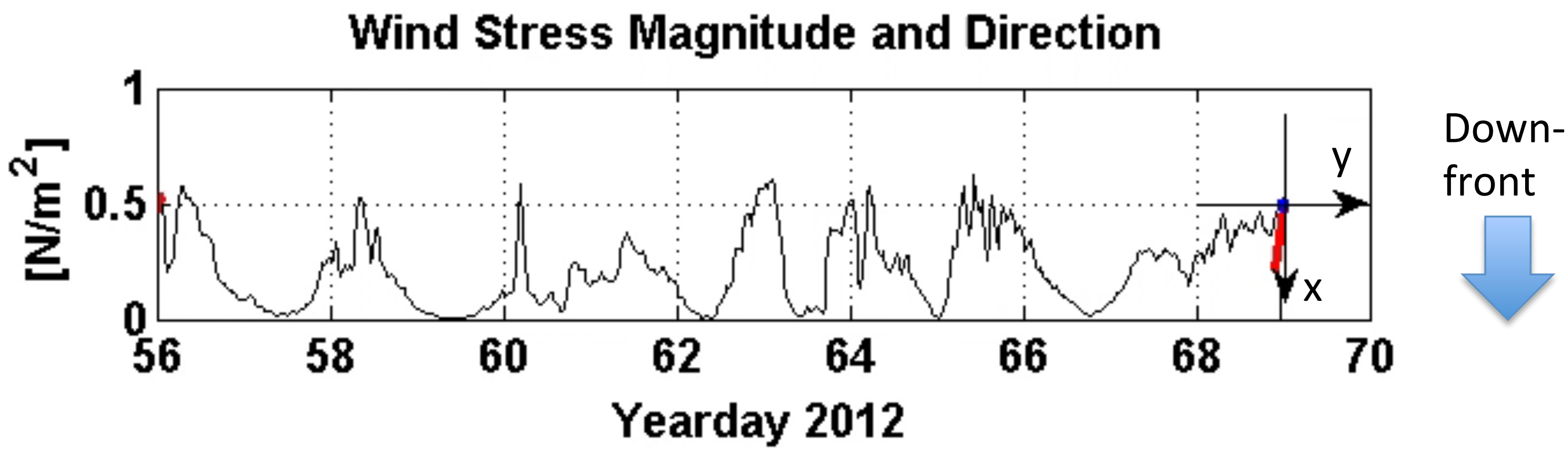
$$Ro_g = \frac{\left|\nabla_h \times \mathbf{u}_g\right|}{f} \sim 0.1-1.0$$

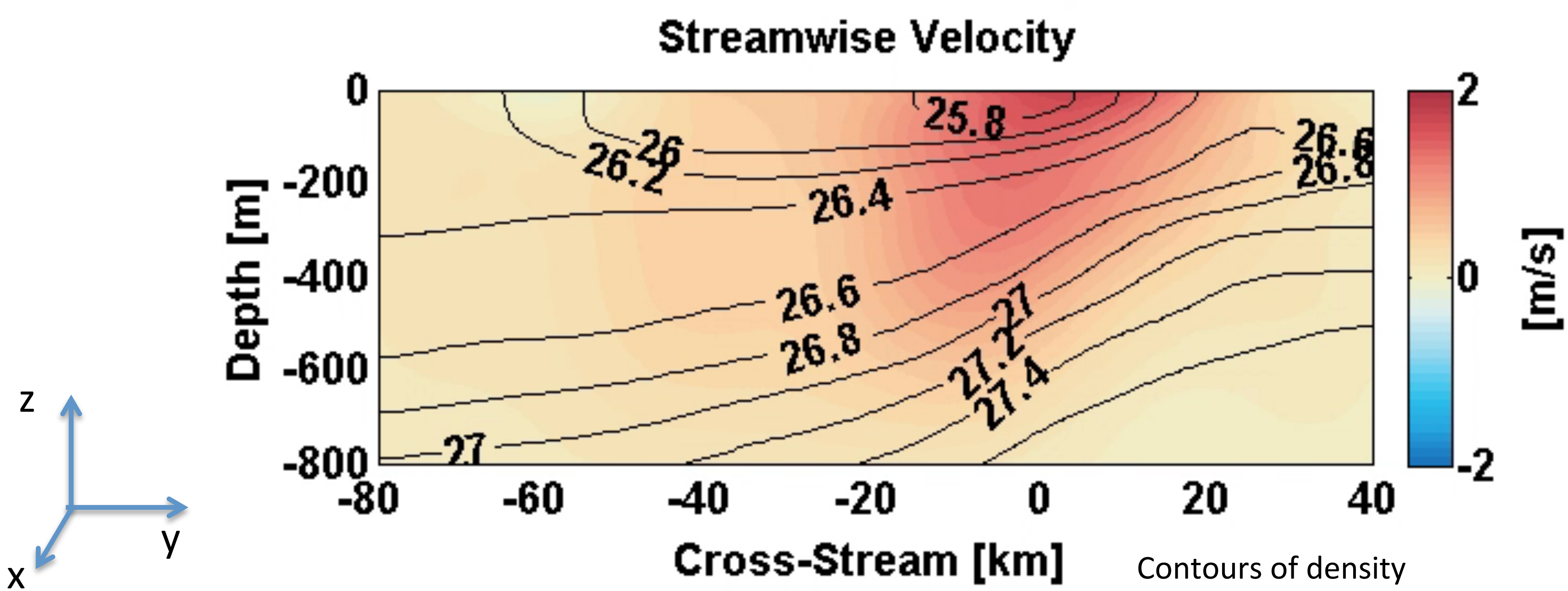


Buoyancy Frequency
Squared

$$N^2 = \partial b / \partial z$$

Near-inertial motions in the Gulf Stream



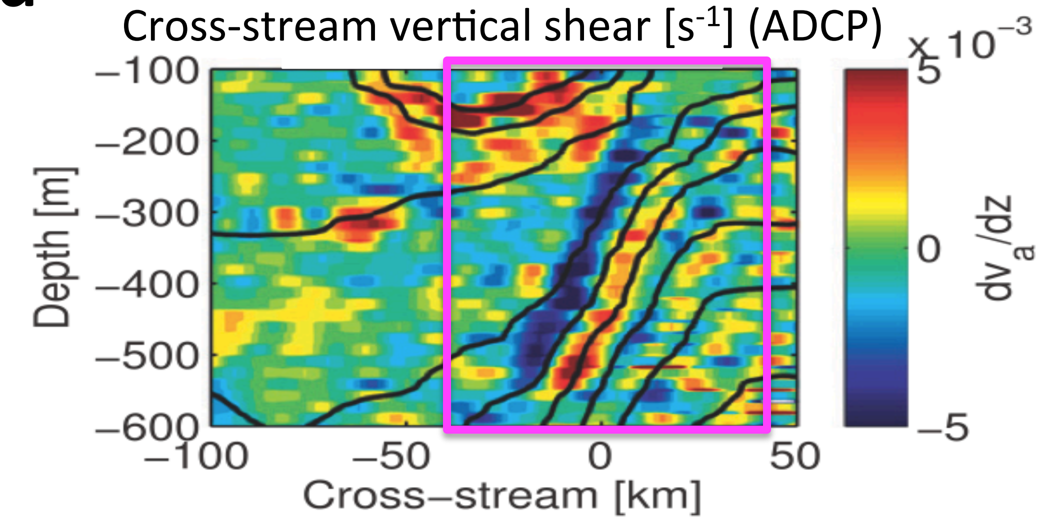


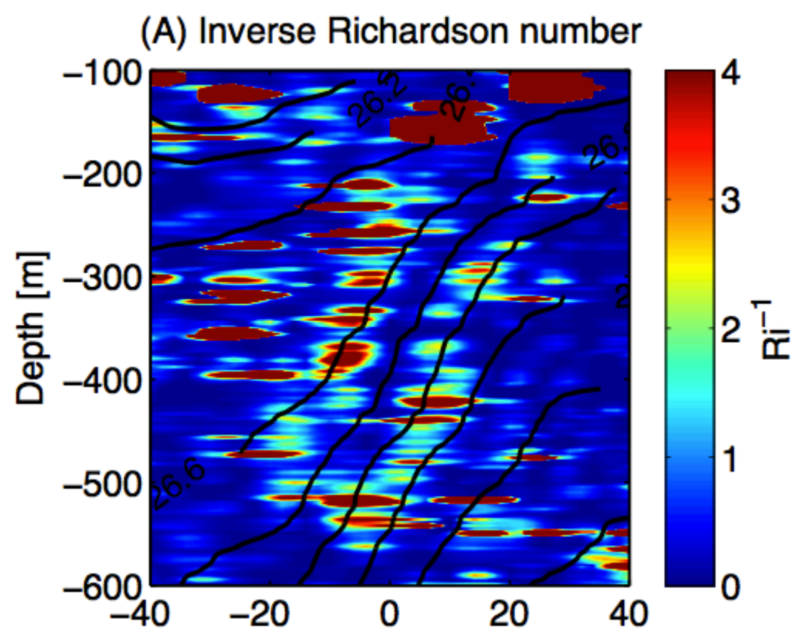
Contours of density anomaly (ρ-1000 kg/m³)

Observations of banded ageostrophic shear in the Gulf Stream

- Parallel to isopycnals, strongest part of the front.
- Banded patterns of high Ri^{-1}
- Energetic turbulence
- Qualitatively consistent with simulations

***Feb. 2007** section at 66° W*





Question for today

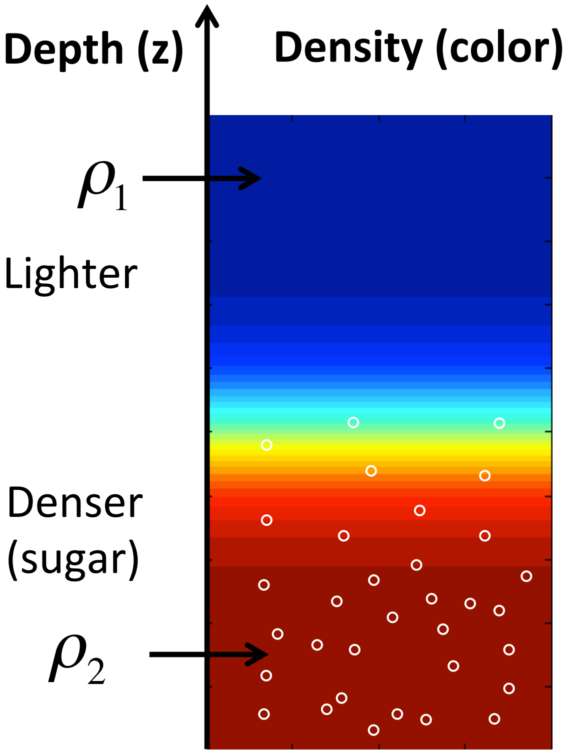
- Wind-forced near-inertial KE develops small horizontal scales over a time scale ~ 1 day and propagates downward as internal inertia-gravity waves.

- **How is the physics of near-inertial internal waves modified by the presence of the strong front like the Gulf Stream?**

Outline

- Lagrangian interpretation of internal waves in rotating, stratified fluids
  - Buoyancy oscillations
  - Inertial oscillations and absolute momentum
  - Inertia-buoyancy oscillations
  - Propagation of internal wave energy
- Near-inertial waves propagating across a geostrophic flow in thermal wind balance
  - Absolute momentum and buoyancy conservation
  - When are symmetric disturbances stable oscillations?
  - Mean flow modification inertial-buoyancy oscillations
  - Propagation of internal wave energy across a strongly baroclinic mean flow
- Interpreting observations in the winter Gulf Stream

Buoyancy oscillations in a density-stratified fluid

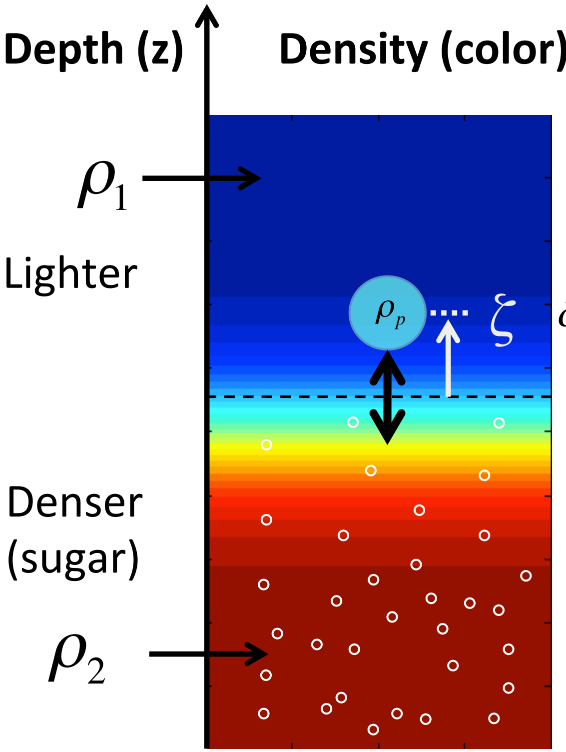


$$\rho_1 < \rho_2$$

Initial Hydrostatic Balance

$$\frac{\partial p}{\partial z} = -g\rho(z)$$

Buoyancy oscillations in a density-stratified fluid



Small adiabatic displacement

$$\delta\rho = \rho_p - \langle\rho\rangle(z_e+\zeta)$$

equilibrium depth

z_e, $\zeta=0$

$$\rho_1 < \rho_2$$

Initial Hydrostatic Balance

$$\frac{\partial p}{\partial z} = -g\rho(z)$$

Conservation Law:

Conservation of density

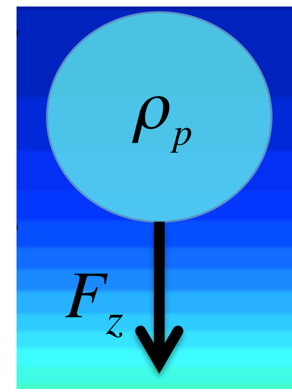
$$\frac{D\rho_p}{Dt} = 0$$

Restoring Force:

Buoyancy

$$F_z = b = -g\frac{\delta\rho}{\rho} = -g\frac{\rho_p - \langle\rho\rangle(z_e+\zeta)}{\langle\rho\rangle(z_e)}$$

$$\approx +\zeta\frac{g}{\rho}\frac{\partial\langle\rho\rangle(z)}{\partial z} = -\zeta N^2$$



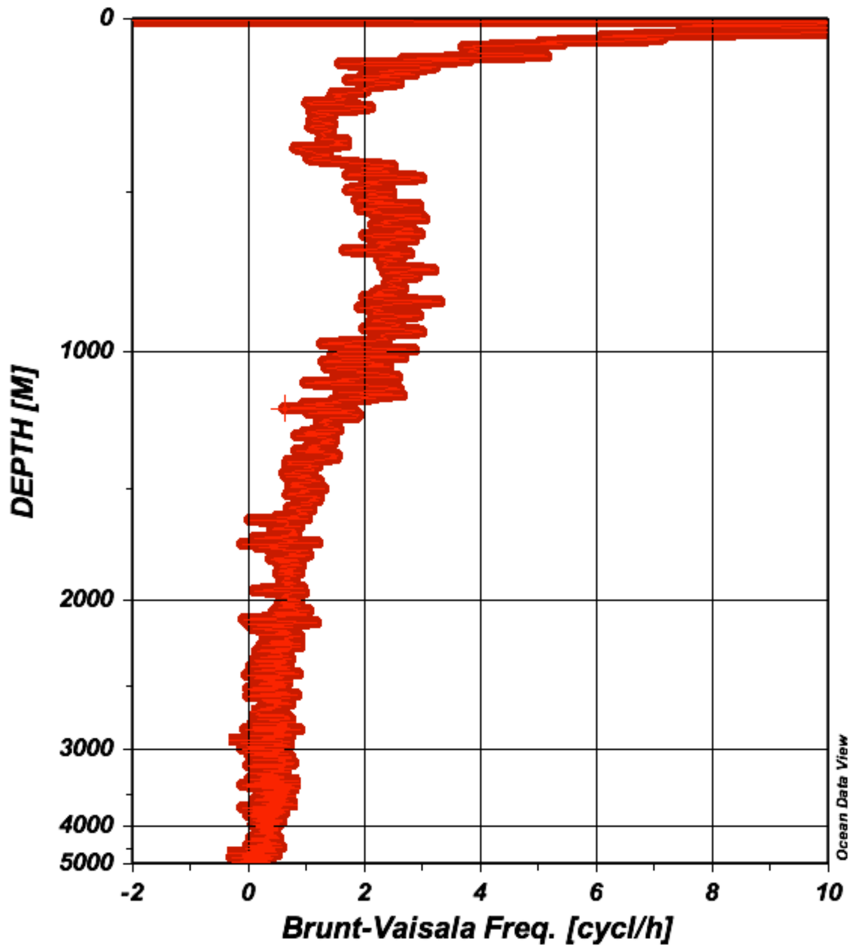
Force balance	$\frac{D^2\zeta}{Dt^2} = \frac{Dw}{Dt} = b = -\zeta N^2$
Conservation of energy	$\frac{D}{Dt}\left(w^2 + \zeta^2 N^2\right) = 0$

Assuming the parcel adjusts instantantly to the background pressure and that external frictional and diabatic effects are negligible

Buoyancy oscillations in a density-stratified fluid

Typical oceanic vertical profile of

$$N = \sqrt{\partial b / \partial z}$$

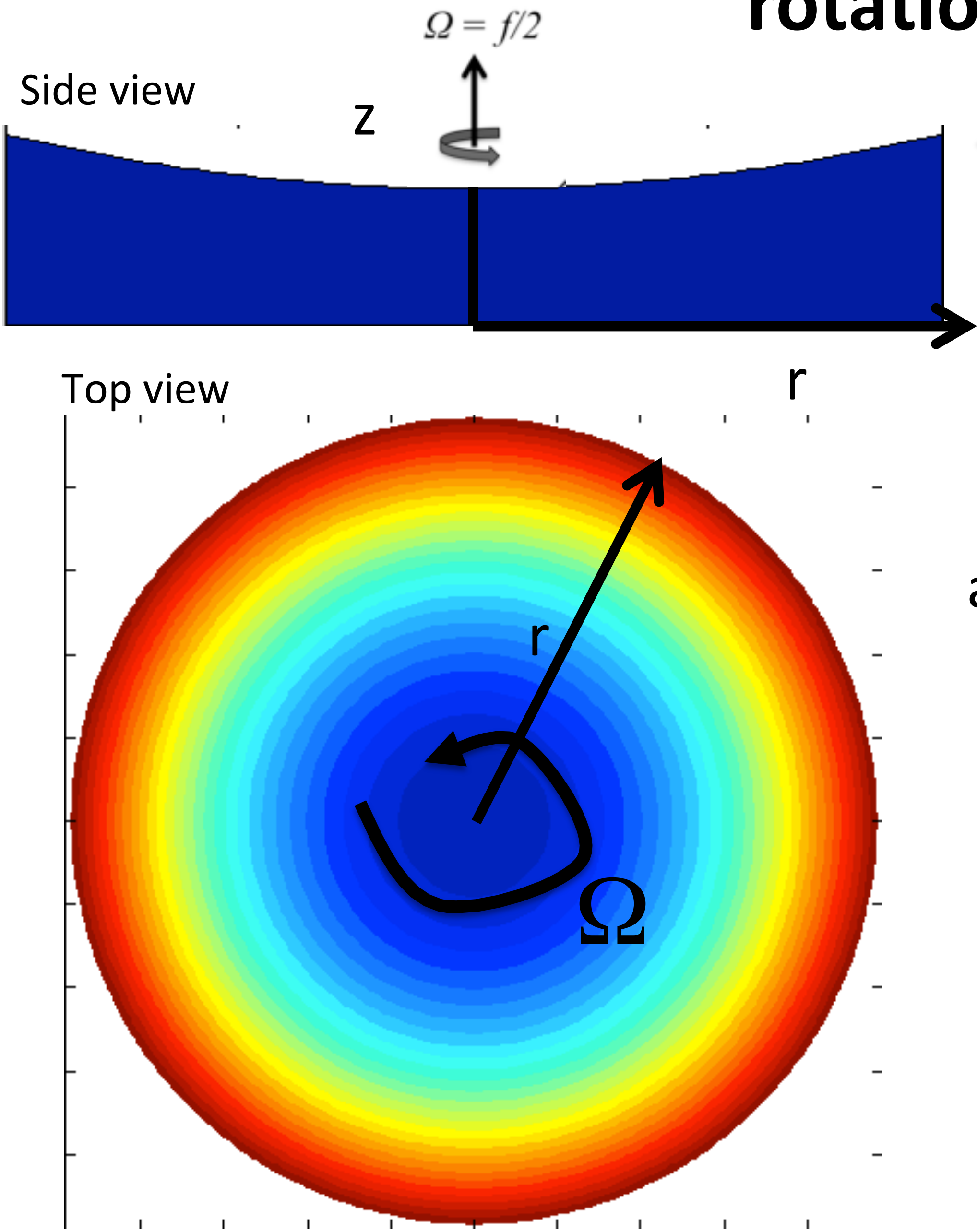


Buoyancy frequency

$$N = \sqrt{-\frac{g}{\rho_0}\frac{\partial \rho}{\partial z}}$$

~ a few minutes in ocean

Inertial oscillations in a fluid disk in solid body rotation



Free surface height (h)

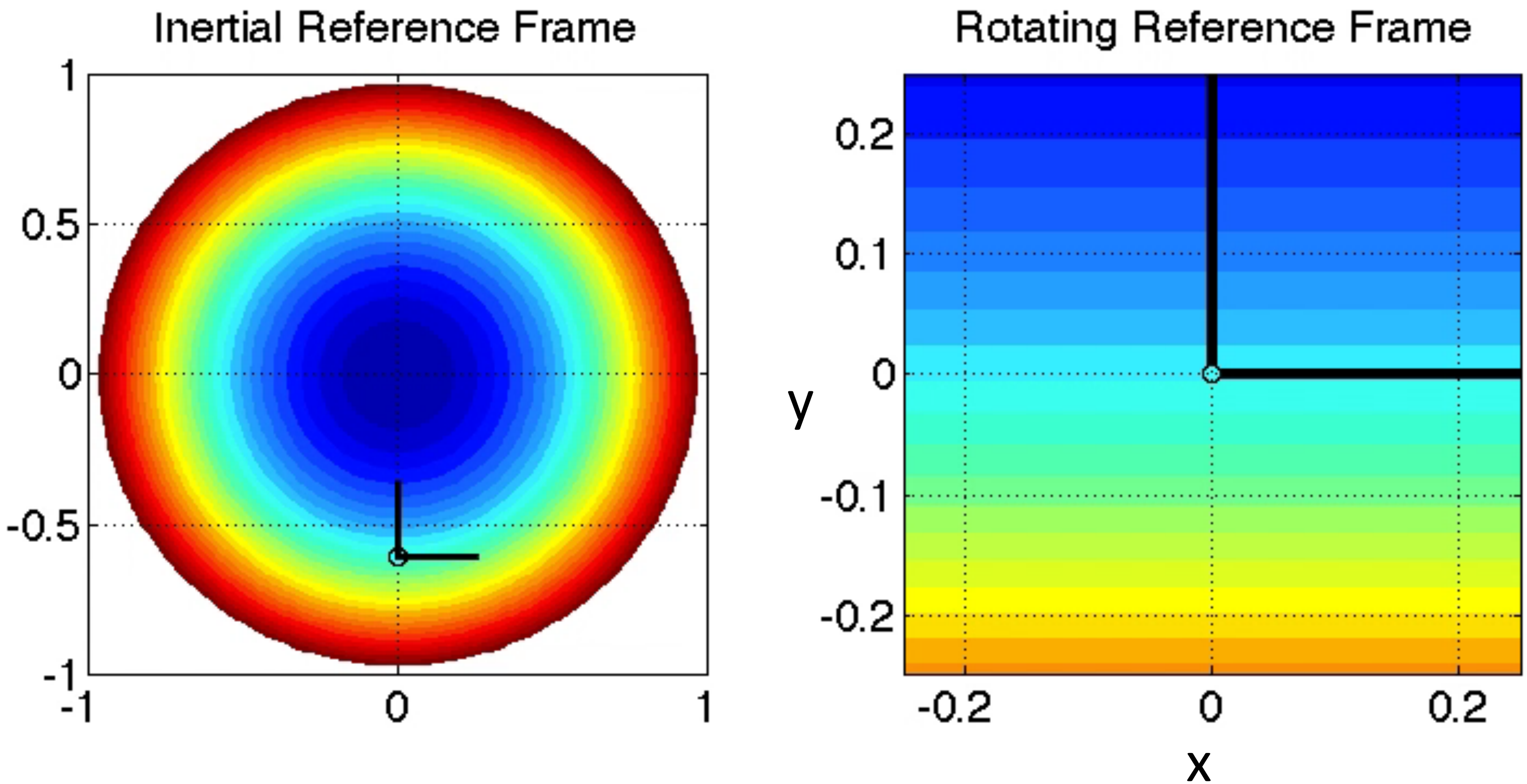
Initial Cyclostrophic balance

$$-\Omega^2 r = -g \frac{\partial h}{\partial r}$$

Free surface height/ angular momentum distribution (color)

$$h = H_0 + \frac{\Omega^2 r^2}{g}, L = \Omega r^2$$

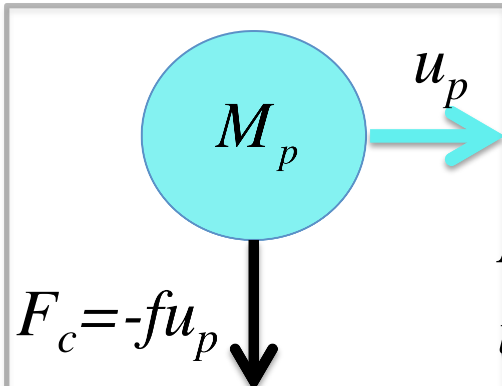
Inertial oscillations in a fluid disk in solid body rotation



Inertial oscillations in a fluid disk in solid body rotation

Conservation Law:
Conservation of absolute momentum

Restoring Force:
Coriolis force



Definition

$$M = u - fy \qquad \langle M \rangle = -fy$$

$$\frac{Du}{Dt} - fv = 0 \Leftrightarrow \frac{DM}{Dt} = 0$$

$$M_p = u_p - f(y_e + \eta) = -fy_e$$

$$u_p(t) = f\eta(t) = \delta M = M_p - \langle M \rangle (y_e + \eta)$$

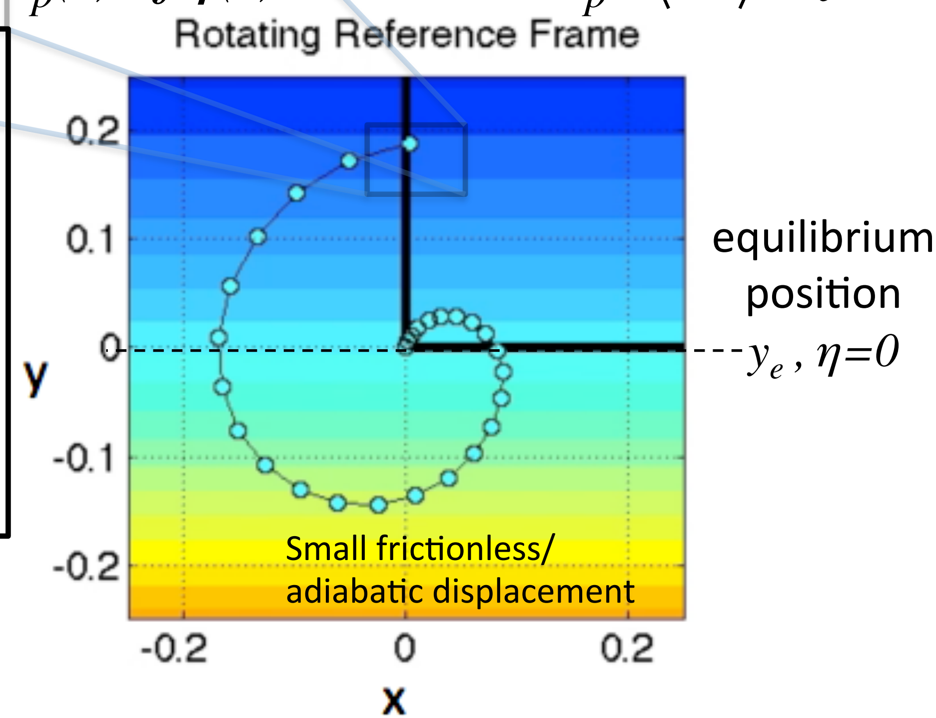
Force balance

$$\frac{D^2\eta}{Dt^2} = \frac{Dv}{Dt} = F_c = -f\delta M \approx \eta f \frac{\partial \langle M \rangle}{\partial y} \approx -\eta f^2$$

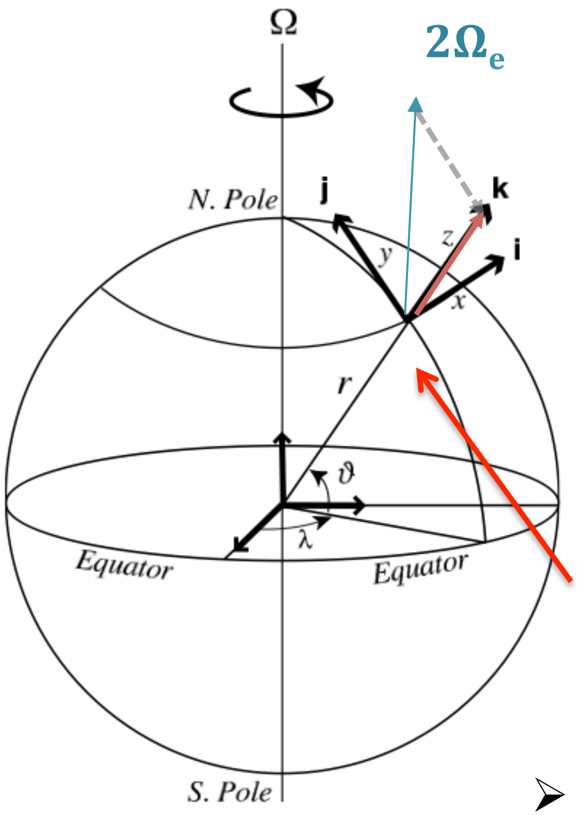
Conservation of energy

$$\frac{D}{Dt}\left(v^2 + f^2\eta^2\right) = \frac{D}{Dt}\left(v^2 + u^2\right) = 0$$

Assuming the parcel adjusts instantantly to the background pressure and that external frictional and diabatic effects are negligible



Small inertial oscillations on a sphere



"Coriolis frequency" or "Inertial" frequency

$$f = 2\Omega_e \sin(latitude)$$

12 hours (at the poles) approaching infinity at the equator

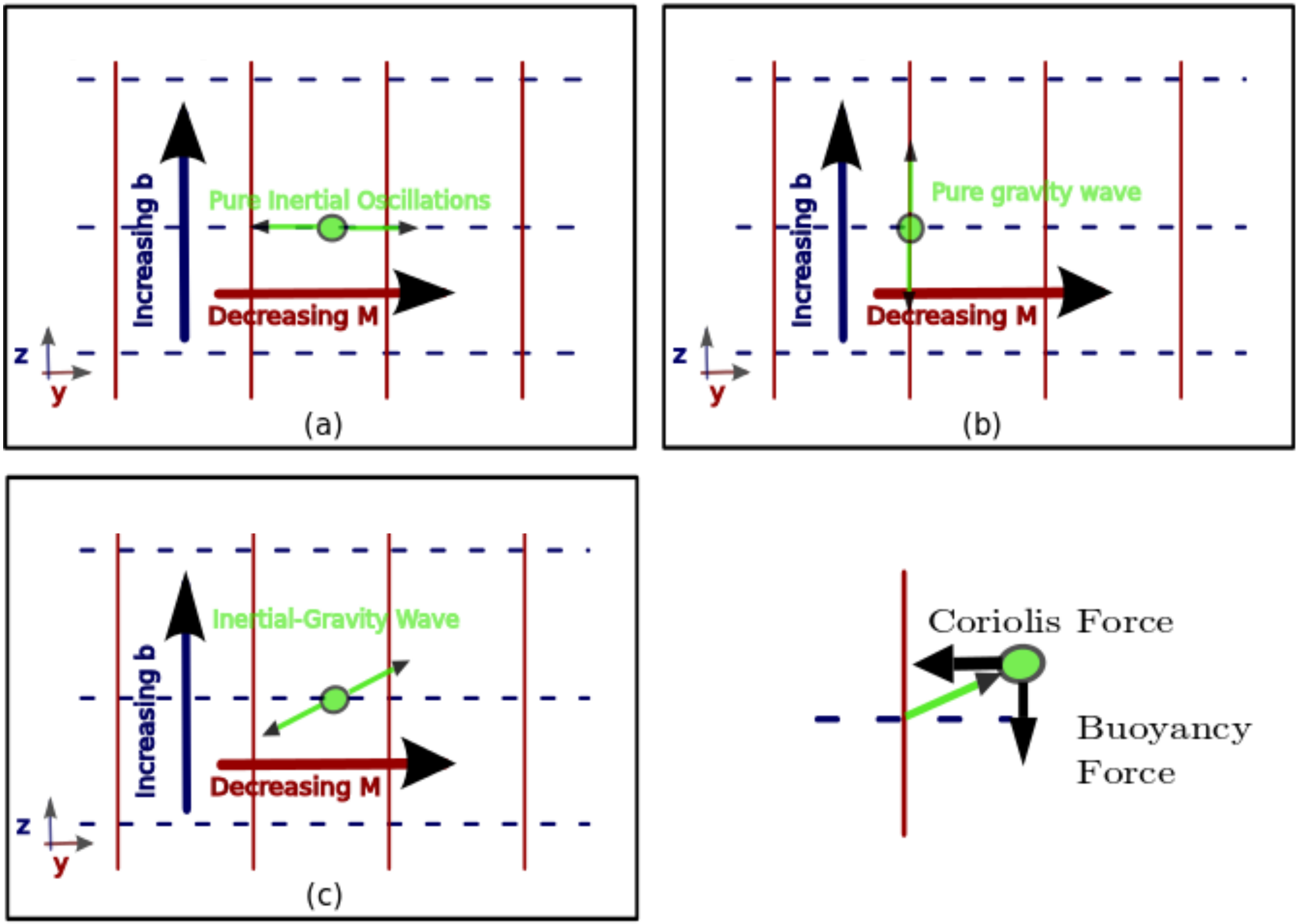
Traditional local tangent plane approximation on a sphere (constant *f*)

- Ignore Coriolis forces that compete with buoyancy force

$$\mathbf{f} = (0, 0, 2\Omega_e \sin(\theta)) \text{ where } \Omega_e \approx 7.3 \times 10^{-5} \text{ s}^{-1}$$

$$\mathbf{f} \times \mathbf{u} = (-fv, fu, 0)$$

Inertia-buoyancy oscillations in a rotating stratified fluid



Inertia-buoyancy oscillations in a rotating stratified fluid

$$\frac{D^2|\vec{\eta}|}{Dt^2} = F_{|\vec{\eta}|} = F_y \cos\theta + F_z \sin\theta$$

Force balance

$$F_y = -fu_p = f\nabla\langle M\rangle \cdot \vec{\eta} = -f^2\eta = -f^2|\eta|\cos\theta$$

$$F_z = b_p = -\nabla\langle b\rangle \cdot \vec{\eta} = -N^2\zeta = -N^2|\eta|\sin\theta$$

Conservation of absolute momentum and buoyancy

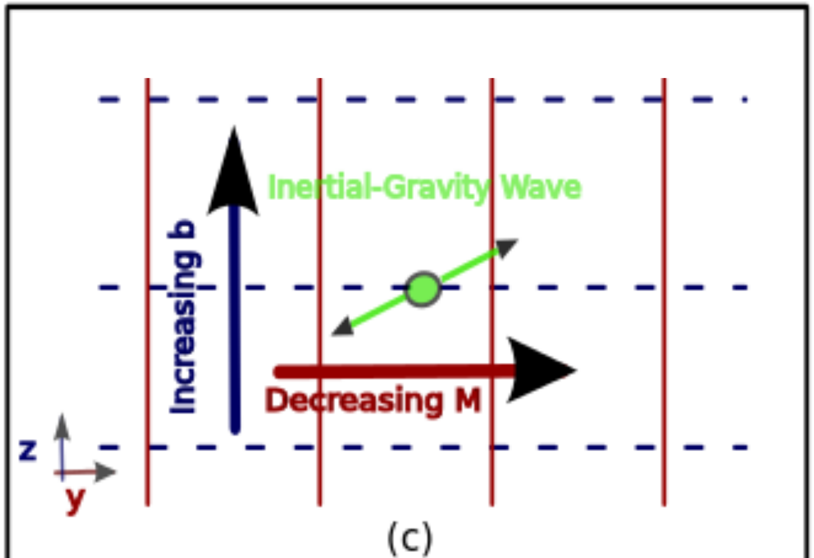
$$\frac{D^2|\vec{\eta}|}{Dt^2} = -|\eta|\omega^2$$

$$\boxed{\omega^2 = \left(f^2\cos^2\theta + N^2\sin^2\theta\right)}$$

Dispersion relation

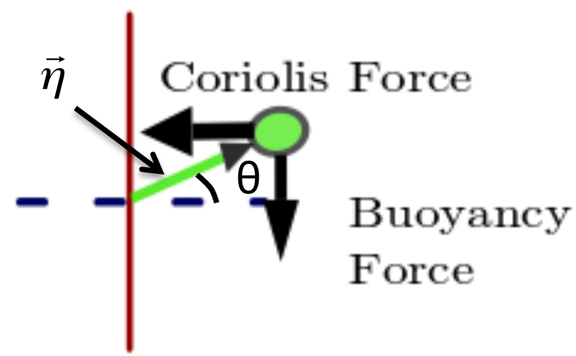
$$\frac{D}{Dt}\left(w^2 + v^2 + f^2\eta^2 + N^2\zeta^2\right) = \frac{D}{Dt}\left(w^2 + v^2 + u^2 + b^2/N^2\right) = 0$$

Conservation of energy



(c)

Coriolis and buoyancy are restoring forces



Inertia-gravity waves $f < \omega < N$

Linearized governing equations

$$\frac{\partial u}{\partial t} - fv = 0$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b$$

$$\frac{\partial b}{\partial t} + wN^2 = 0$$

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Wave energy conservation

$$\frac{\partial E}{\partial t} + \nabla_{y,z} \cdot (p\mathbf{u}) = 0$$

$$E = \rho_0 \frac{v^2 + w^2 + f^2\eta^2 + N^2\zeta^2}{2}$$

$$= \rho_0 \frac{v^2 + w^2 + u^2 + b^2 / N^2}{2}$$

$$\boxed{\left(\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}\right] + f^2\frac{\partial^2}{\partial z^2} + N^2\frac{\partial^2}{\partial y^2}\right)\psi = 0}$$

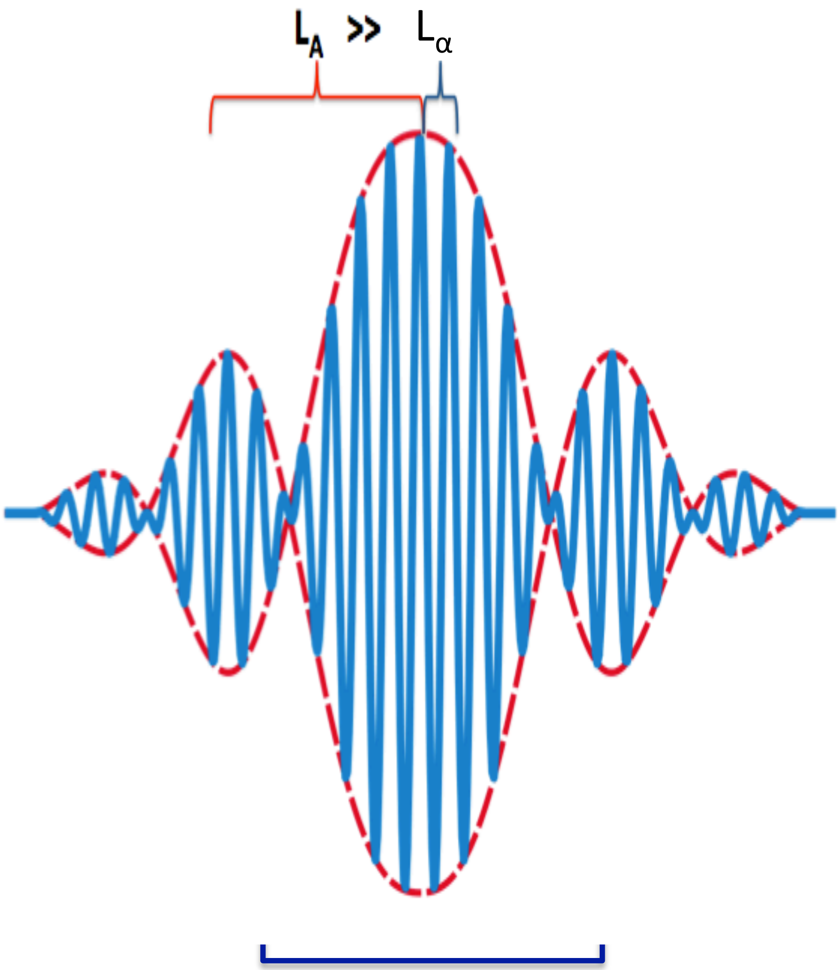
- Parcel arguments cannot describe the wavelike properties
- Energy at a given frequency propagates at a fixed angle from horizontal in constant N, f

http://dennou.gaia.h.kyoto-u.ac.jp/library/gfd_exp/exp_e/exp/iw/1/res.htm

Inertia-gravity waves $f < \omega < N$

A Plane Wave Solution

$$\mathbf{k}_h = 0, \omega = f$$



$$\boxed{\left(\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial y^2}\right]+f^2\frac{\partial^2}{\partial z^2}+N^2\frac{\partial^2}{\partial y^2}\right)\psi = 0}$$

$$\psi \propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \qquad \frac{l}{m} = -\lambda_{\pm} = \pm\sqrt{\frac{\omega^2 - f^2}{N^2-\omega^2}}$$

$$\frac{|\mathbf{k}_h|}{|\mathbf{k}|} = \cos\phi \qquad \frac{m}{|\mathbf{k}|} = \sin\phi$$

$$\boxed{\omega^2 = f^2\sin^2\phi + N^2\cos^2\phi}$$

- Pressure gradient force orthogonal to parcel velocities in plane wave solutions -> no energy propagation.
- Energy propagates in slowly-varying plane waves,

$$\psi = Re(\psi_0(x,y,z,t)e^{i(\alpha(x,y,z,t)})$$

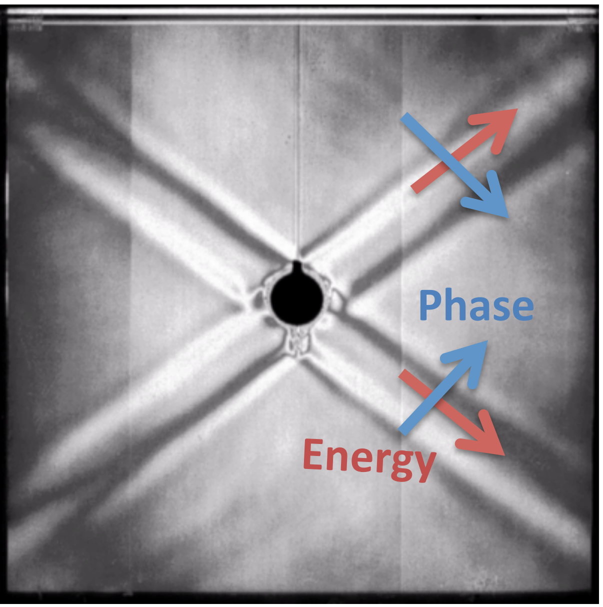
$$\frac{\partial E}{\partial t} + \nabla_{y,z}\cdot(p\mathbf{u}) = 0 \qquad \mathbf{c}_g = \nabla_{l,m}\omega$$

$$\mathbf{F}_a = \mathbf{c}_g\langle E\rangle = p\mathbf{u}_a$$

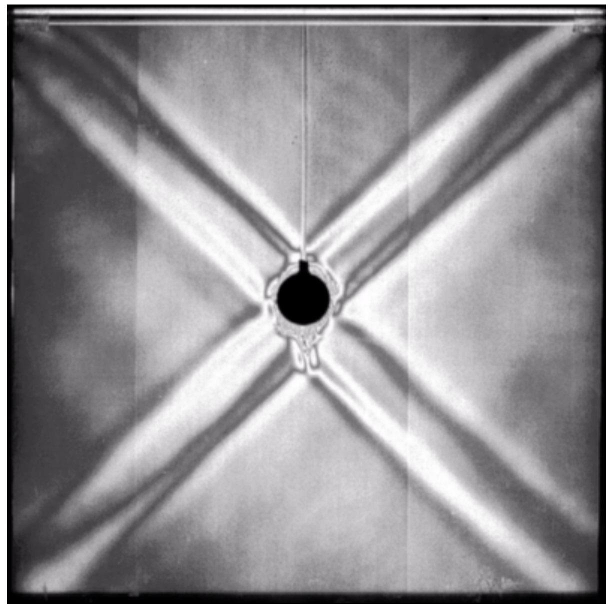
When $f < \omega < N$, wave energy can propagate

$$\omega^2 = f^2 \cos\theta + N^2 \sin\theta$$

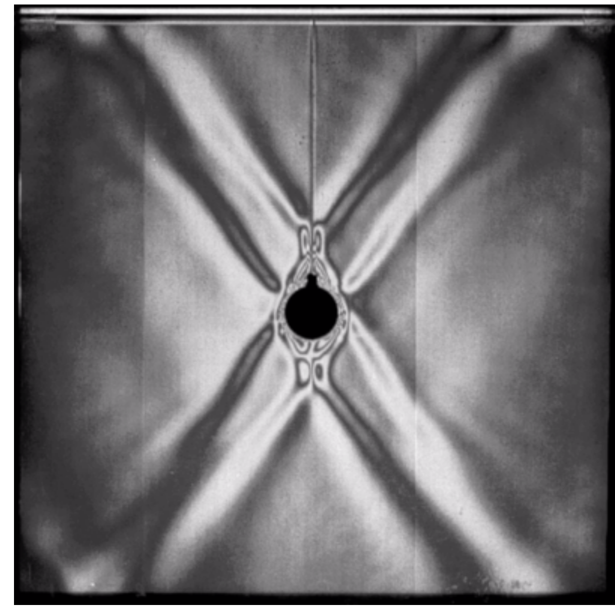
$T_{forcing} = 7$ sec



$T_{forcing} = 6$ sec



$T_{forcing} = 5$ sec



- Phase lines propagate perpendicular to energy propagation
  - Energy propagates at the **group velocity** $\mathbf{c}_g = \nabla_{l,m}\omega$
  - Shallower slope at lower frequencies. $\mathbf{F}_a = \mathbf{c}_g \langle E \rangle = p\mathbf{u}_a$
- Characteristics symmetric about horizontal axis

http://dennou.gaia.h.kyoto-u.ac.jp/library/gfd_exp/exp_e/exp/iw/1/res.htm

Small, unsteady, symmetric perturbations in a steady and symmetric baroclinic geostrophic flow

$$u = u_g + u_a, \quad v = v_a, \quad w = w_a, \quad b = b_g + b_a$$

$$v_a = \frac{\partial \psi}{\partial z}, \quad w_a = -\frac{\partial \psi}{\partial y}$$

Modified wave eqn

$$\boxed{\left(F^2 + \frac{\partial^2}{\partial t^2}\right)\frac{\partial^2 \psi}{\partial z^2} + \boxed{2S^2 \frac{\partial^2 \psi}{\partial z \partial y}} + N^2 \frac{\partial^2 \psi}{\partial y^2} = 0}$$

Classic wave eqn:

$$\boxed{\left(\left[\frac{\partial^2}{\partial t^2} + f^2\right]\frac{\partial^2}{\partial z^2} + N^2 \frac{\partial^2}{\partial y^2}\right)\psi = 0}$$

$$F^2 = f(f - \partial u_g/\partial y), \; S^2 = f\partial u_g/\partial z = -\partial b_g/\partial y,$$
$$N^2 = \partial b_g/\partial z$$

$$\frac{\partial u_a}{\partial t} \boxed{+ v_a \frac{\partial u_g}{\partial y} \boxed{+ w_a \frac{\partial u_g}{\partial z}}} - f v_a = 0,$$

$$\frac{\partial v_a}{\partial t} + f u_a = -\frac{1}{\rho_0}\frac{\partial p_a}{\partial y},$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p_a}{\partial z} + b_a,$$

$$\frac{\partial b_a}{\partial t} \boxed{+ v_a \frac{\partial b_g}{\partial y}} + w_a \frac{\partial b_g}{\partial z} = 0,$$

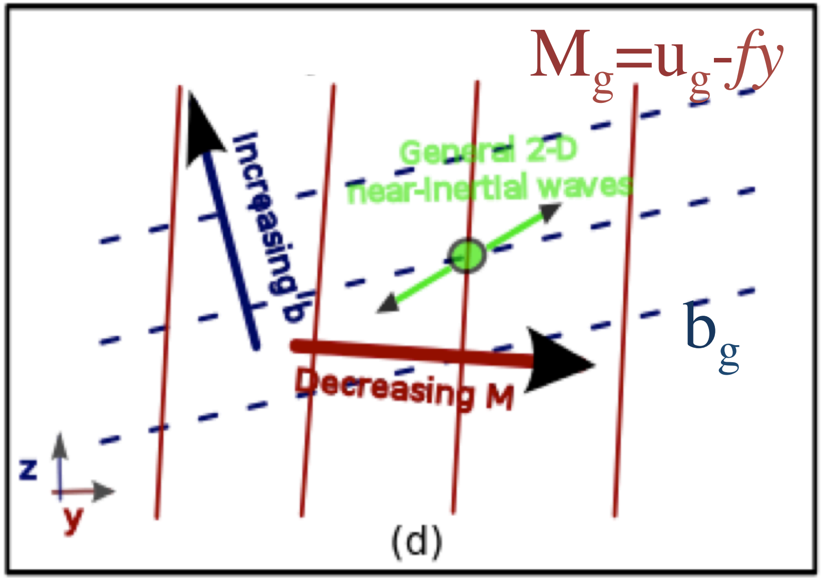
$$\frac{\partial v_a}{\partial y} + \frac{\partial w_a}{\partial z} = 0,$$

$$f\frac{\partial u_g}{\partial z} = -\frac{\partial b_g}{\partial y}$$

[see also Mooers 1975, Kunze 1985, Young and Ben Jelloul 1997, Plougonven and Zeitlin 2005]

Small, unsteady, symmetric perturbations in a steady and symmetric baroclinic geostrophic flow

$Ri_g^{-1} = \left|\partial \mathbf{u}_g / \partial z\right|^2 \Big/ N^2 > 0$ $\quad Ro_g = \dfrac{-\partial u_g / \partial y}{f}$



$$\boxed{\frac{\partial u_a}{\partial t} + v_a \frac{\partial u_g}{\partial y} + w_a \frac{\partial u_g}{\partial z} - f v_a = 0,}$$

$$\boxed{\frac{\partial v_a}{\partial t} + f u_a = -\frac{1}{\rho_0}\frac{\partial p_a}{\partial y},}$$

$$\boxed{0 = -\frac{1}{\rho_0}\frac{\partial p_a}{\partial z} + b_a,}$$

$$\boxed{\frac{\partial b_a}{\partial t} + v_a \frac{\partial b_g}{\partial y} + w_a \frac{\partial b_g}{\partial z} = 0,}$$

$$\frac{\partial v_a}{\partial y} + \frac{\partial w_a}{\partial z} = 0,$$

$$f\frac{\partial u_g}{\partial z} = -\frac{\partial b_g}{\partial y}$$

Two Conservation Laws

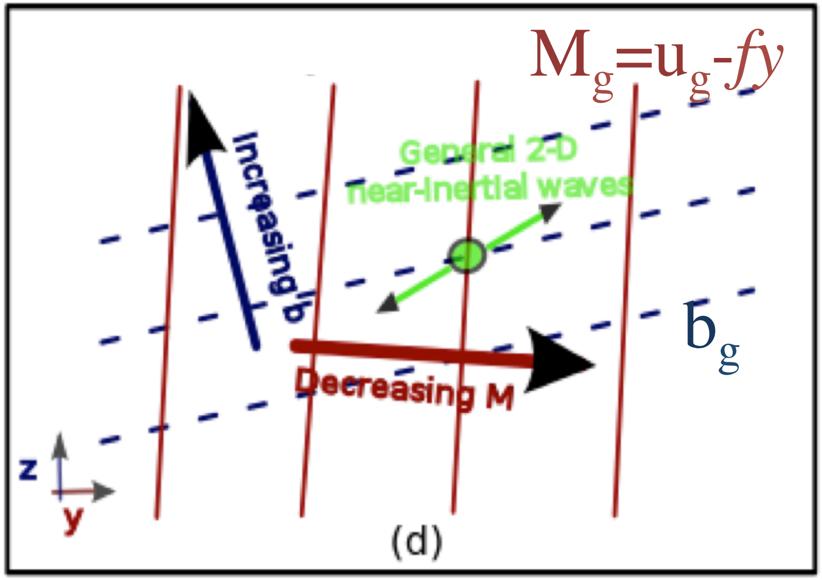
$\dfrac{DM_T}{Dt} = 0$ *Absolute momentum* $M_T = u_a + u_g - f(y_e + \eta) = u_a + M_g$

$\dfrac{Db_T}{Dt} = 0$ *Buoyancy* $b_T = b_a + b_g$

Parcel Displacements $\eta(T) = \int_0^T v\,dt, \zeta(T) = \int_0^T w\,dt$

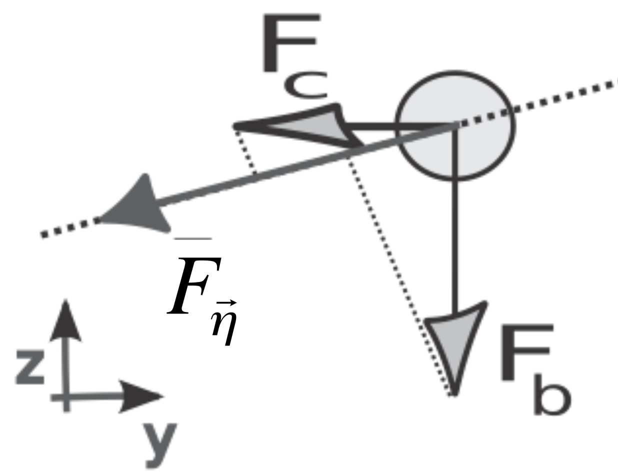
Small, unsteady, symmetric perturbations in a steady and symmetric baroclinic geostrophic flow

$Ri_g^{-1} = \left|\partial \mathbf{u}_g / \partial z\right|^2 / N^2 > 0 \qquad Ro_g = \frac{-\partial u_g / \partial y}{f}$



Stable oscillatory example

Force Diagram



$$F_{\vec{\eta}} = proj_{\vec{\eta}} F_c + proj_{\vec{\eta}} F_b$$

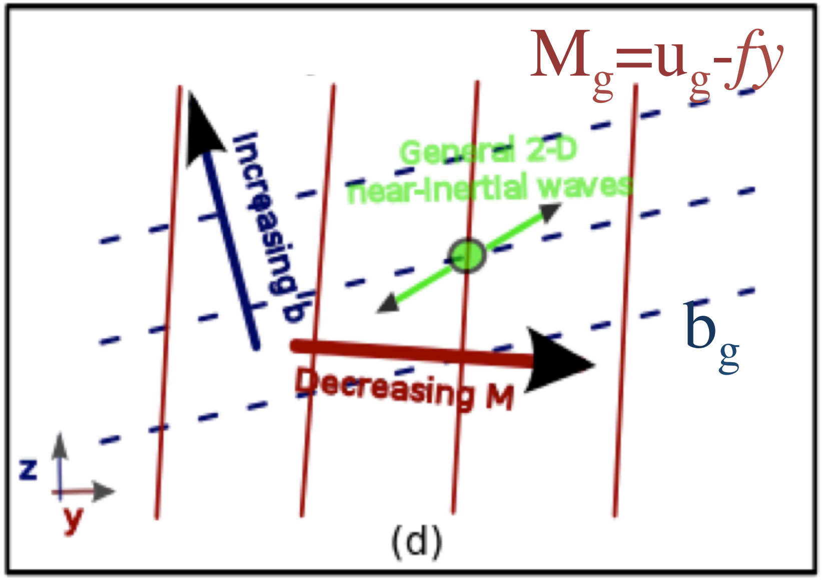
Forces on a parcel in the cross-front plane:

$$F_\eta = \frac{Dv_a}{Dt} = -fu_a = f\nabla M_g \cdot \boldsymbol{\eta} = f\left(\eta \frac{\partial M_g}{\partial y} + \zeta \frac{\partial M_g}{\partial z}\right) = -F^2\eta + S^2\zeta,$$

$$F_\zeta = \frac{Dw_a}{Dt} = b_a = -\nabla b_g \cdot \boldsymbol{\eta} = -\zeta \frac{\partial b_g}{\partial z} - \eta \frac{\partial b_g}{\partial y} = -N^2\zeta + S^2\eta.$$

Small, unsteady, symmetric perturbations in a steady and symmetric baroclinic geostrophic flow

$Ri_g^{-1} = \left|\partial \mathbf{u}_g / \partial z\right|^2 / N^2 > 0$ $\qquad Ro_g = \frac{-\partial u_g / \partial y}{f}$



Ertel PV: $q = \frac{\partial(b_g, M_g)}{\partial(y, z)}$

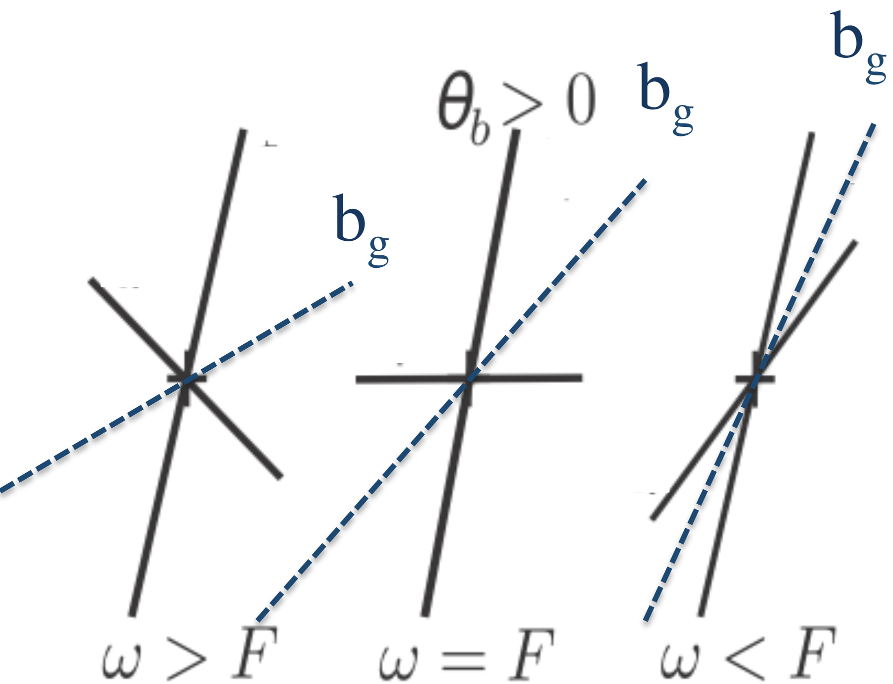
Inertia-gravity waves when $\mathbf{b_g}$-surfaces are shallower than $\mathbf{M_g}$-surfaces ($q>0$ in NH), admits only real frequencies/complex growth rates.

Symmetric Instabilities when $\mathbf{b_g}$-surfaces are steeper than $\mathbf{M_g}$-surfaces ($q<0$ in NH), admits complex frequencies/real growth rates [Hoskins 1974].

$$\frac{D^2 |\eta|}{Dt^2} = -|\eta|\left(F^2\cos^2(\theta) - 2S^2\sin(\theta)\cos(\theta) + N^2\sin^2(\theta)\right)$$

$$\begin{aligned}\omega &= \sqrt{F^2\cos^2(\theta) - 2S^2\sin(\theta)\cos(\theta) + N^2\sin^2(\theta)} \\ &\approx \sqrt{F^2 - 2S^2\theta + N^2\theta^2}.\end{aligned}$$

Inertia-gravity waves when $\omega^2 > max(0 , fq/N^2)$



Wave equation is hyperbolic

$$\left[\left(F^2-\omega^2\right)\frac{\partial^2}{\partial z^2}+2S^2\frac{\partial^2}{\partial y\partial z}+N^2\frac{\partial^2}{\partial y^2}\right]\psi=0$$

- Two characteristic slopes for a given ω,
- Symmetric about isopycnals

$$\lambda_\pm=-\frac{S^2\pm\sqrt{S^4-N^2(F^2-\omega^2)}}{N^2}$$

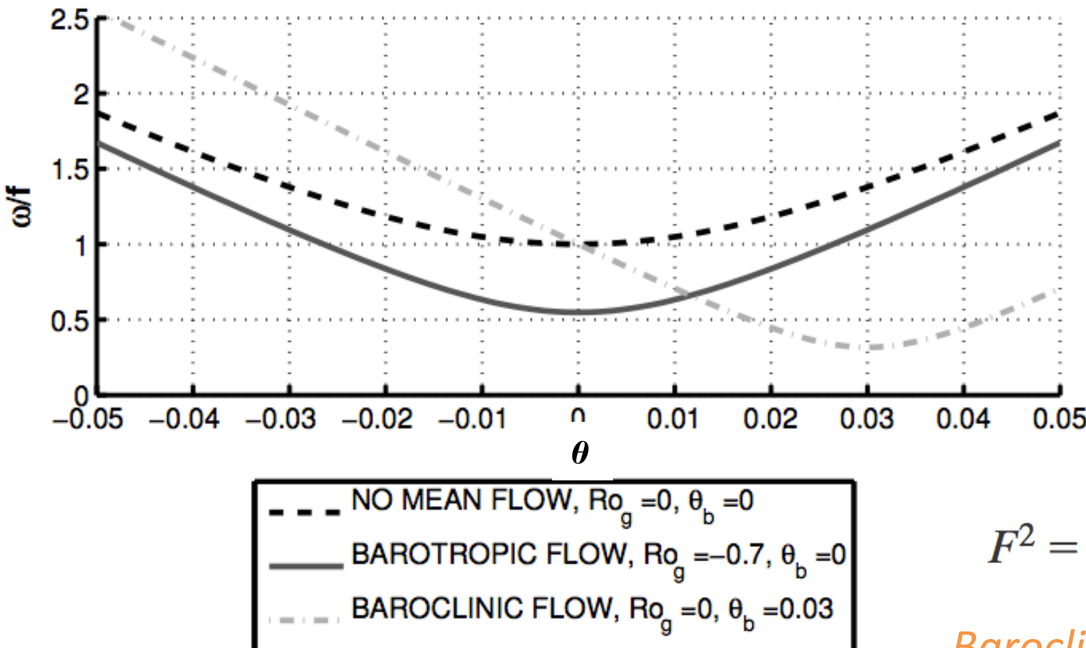
$$\boxed{=\quad \tan(\theta_b)\pm\sqrt{\frac{\omega^2-\omega^2_{\min}}{N^2}}}$$

Minimum frequency inertial oscillations parallel to isopycnals

$$\omega_{\min}=\sqrt{F^2-S^4/N^2}=\sqrt{\frac{fq}{N^2}}$$

$$F^2=f(f-\partial u_g/\partial y),\ S^2=f\partial u_g/\partial z=-\partial b_g/\partial y,$$
$$N^2=\partial b_g/\partial z$$

Geostrophic flows modify the dispersion relation



$$\boxed{\omega \approx \sqrt{F^2 - 2S^2\theta + N^2\theta^2}}$$

$$N^2 = \partial b_g / \partial z$$

$$F^2 = f(f - \partial u_g/\partial y),\ S^2 = f\partial u_g/\partial z = -\partial b_g/\partial y,$$

Baroclinicity reduces minimum frequency

$$\omega_{\min} = \sqrt{F^2 - S^4/N^2} = \sqrt{\frac{fq}{N^2}} = f\sqrt{1 + Ro_g - \boxed{Ri_g^{-1}}}$$

where $Ro_g = -\partial u_g/\partial y/f$ and $Ri_g = f^2N^2/S^4$

q is a potential vorticity

Minimum frequency inertial oscillations

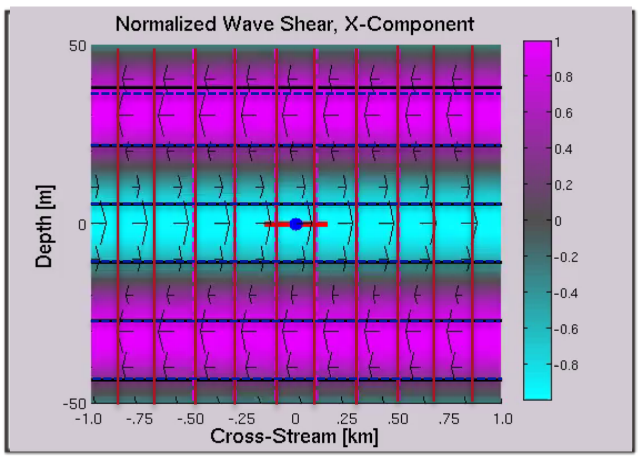
$$\omega_{\min} = \sqrt{F^2 - S^4/N^2} = \sqrt{\frac{fq}{N^2}} = f\sqrt{1 + Ro_g - Ri_g^{-1}}$$

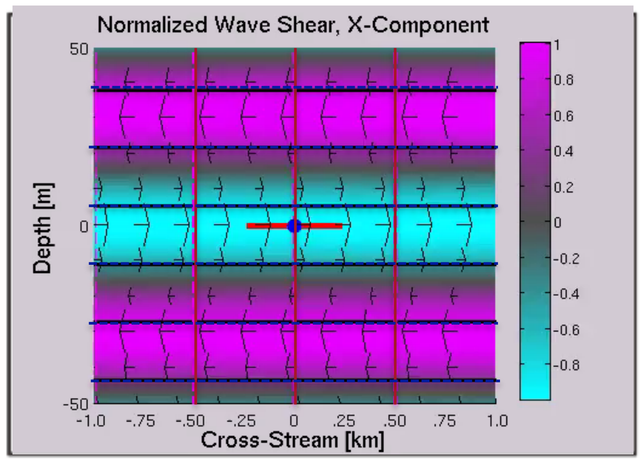
Characteristic slope and parcel oscillations parallel to isopycnals, $s_b = S^2/N^2$

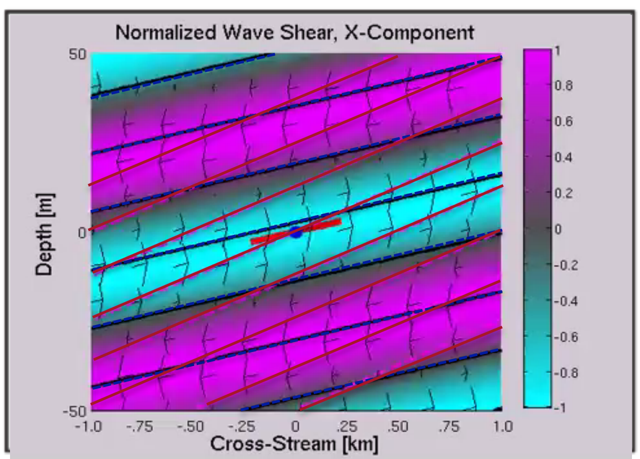
Physics Governed by Conservation of Absolute Momentum

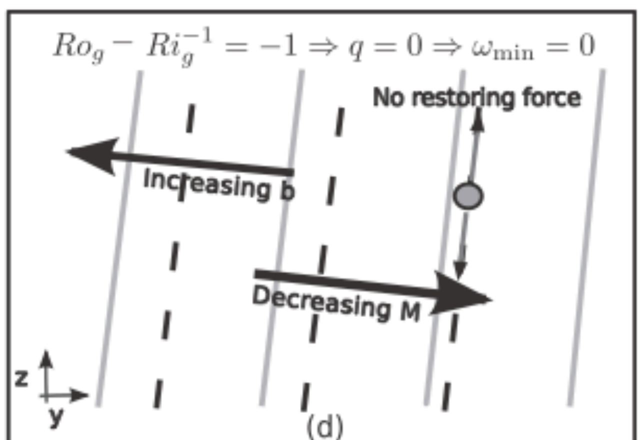
$$\frac{DM_T}{Dt} = 0 \quad M_T = u_a + u_g - f(y_e + \eta) \qquad \frac{D\eta}{Dt} = v$$

Minimum frequency depends on gradient of M_g on isopycnals.









Insight from 2-D PV (q):

$$q = \frac{\partial(b_g, M_g)}{\partial(y, z)}$$

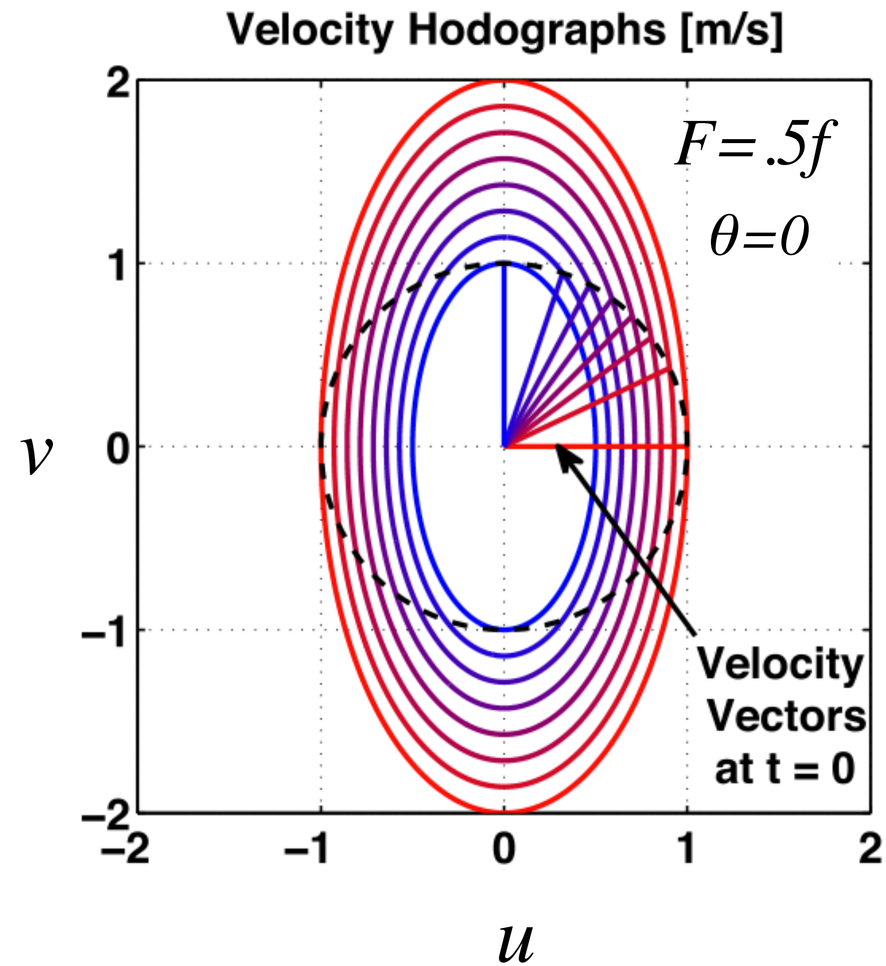
$$\boxed{\omega_{\min} = \left[\frac{f}{N^2}\frac{\partial(b_g, M_g)}{\partial(y, z)}\right]^{1/2} \qquad \omega_{\min} = \{S^2[\tan(\theta_M) - \tan(\theta_b)]\}^{1/2}}$$

Polarization of horizontal velocity

$$u_a = \frac{i}{f\omega}\left(F^2 - S^2\,\theta\right) v_a$$

$$F = f(1+Ro_g)^{1/2}$$

$$u_a/v_a = (1+Ro_g)^{1/2}$$



$F^2 = f(f - \partial u_g/\partial y),\ S^2 = f\partial u_g/\partial z = -\partial b_g/\partial y,$

[see Whitt and Thomas 2015 JPO]

Polarization of horizontal velocity

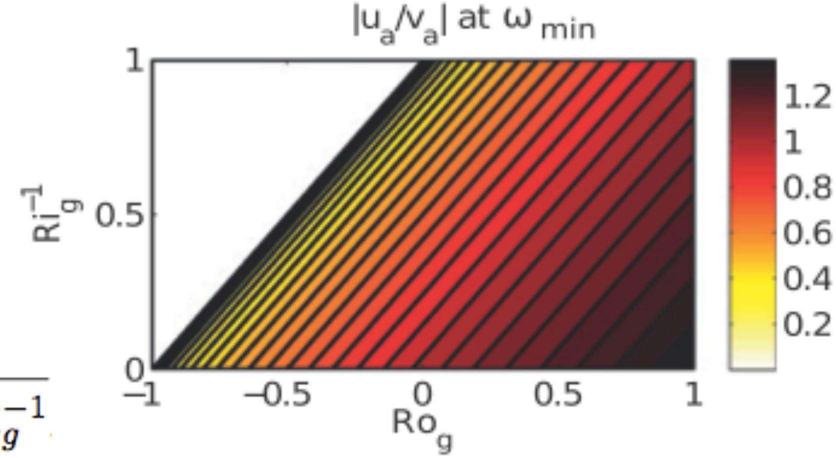
$$u_a = \frac{i}{f\omega}\left(F^2 - S^2\,\theta\right) v_a$$

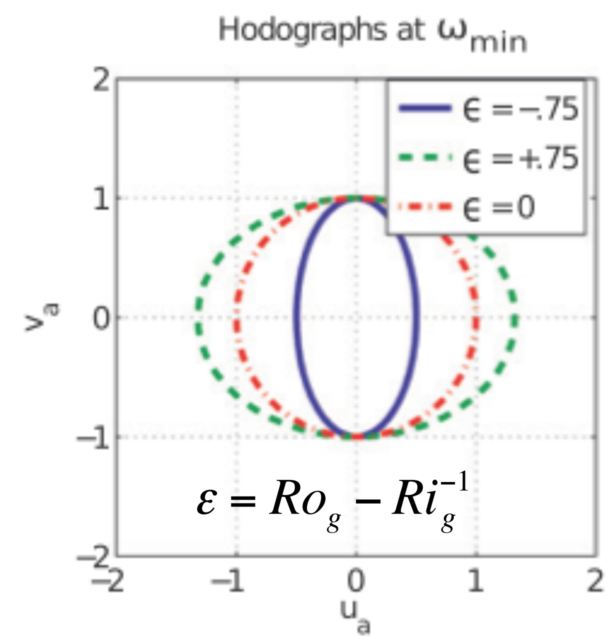
Minimum frequency inertial oscillations parallel to

$$\omega_{\min} = \sqrt{F^2 - S^4/N^2} = \sqrt{\frac{fq}{N^2}} = f\sqrt{1 + Ro_g - Ri_g^{-1}}$$

have elliptical hodographs in a geostrophic flow:

$$u_a = i v_a \sqrt{1 + \mathrm{Ro}_g - \mathrm{Ri}_g^{-1}}$$





$$F^2 = f(f - \partial u_g/\partial y),\ S^2 = f\partial u_g/\partial z = -\partial b_g/\partial y,$$

Linearized wave energy is conserved in the absence of forcing, diabatic, and viscous effects

Integrate buoyancy and momentum conservation laws

$$u_a = -\eta \frac{\partial M_g}{\partial y} - \zeta \frac{\partial M_g}{\partial z} \qquad b_a = -\eta \frac{\partial b_g}{\partial y} - \zeta \frac{\partial b_g}{\partial z}$$

$$\frac{\partial (v_a^2/2)}{\partial t} + f u_a v_a - b_a w_a = -\frac{1}{\rho_0}\left(\frac{\partial p_a v_a}{\partial y} + \frac{\partial p_a w_a}{\partial z} \right)$$

Form an “energy” equation

$$\frac{\partial (v_a^2/2)}{\partial t} + (F^2\eta - S^2\zeta) v_a - (S^2\eta - N^2\zeta) w_a = -\frac{1}{\rho_0}\left(\frac{\partial p_a v_a}{\partial y} + \frac{\partial p_a w_a}{\partial z} \right)$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left[v_a^2 + F^2\eta^2 - 2S^2\zeta\eta + N^2\zeta^2 \right] = -\frac{1}{\rho_0}\left(\frac{\partial p_a v_a}{\partial y} + \frac{\partial p_a w_a}{\partial z} \right)$$

Wave activity conservation law

$$\boxed{\begin{gathered} A = \rho_0 \frac{\eta_t^2 + F^2\eta^2 - 2S^2\eta\zeta + N^2\zeta^2}{2} \\[2ex] \frac{\partial A}{\partial t} + \nabla_{y,z} \cdot \left(p\mathbf{u}_a \right) = 0 \end{gathered}}$$

Equivalent to usual perturbation energy $\langle A \rangle_T = \langle E \rangle_T = 1/2(u^2 + v^2 + b^2/N^2)$

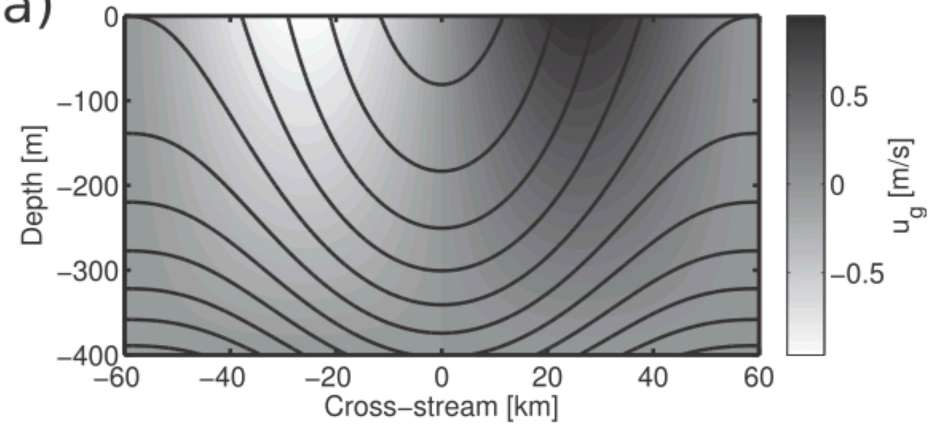
<u>when integrated over an integer number of wave periods</u> $T=2\pi/\omega$ in SHM

Cross-stream propagation of sub-inertial waves in a spatially-variable geostrophic flow

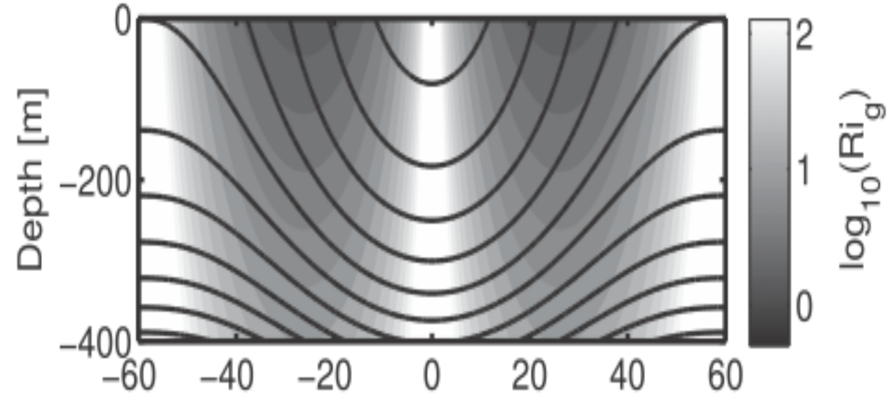
An idealized domain

Geostrophic Velocity

a)

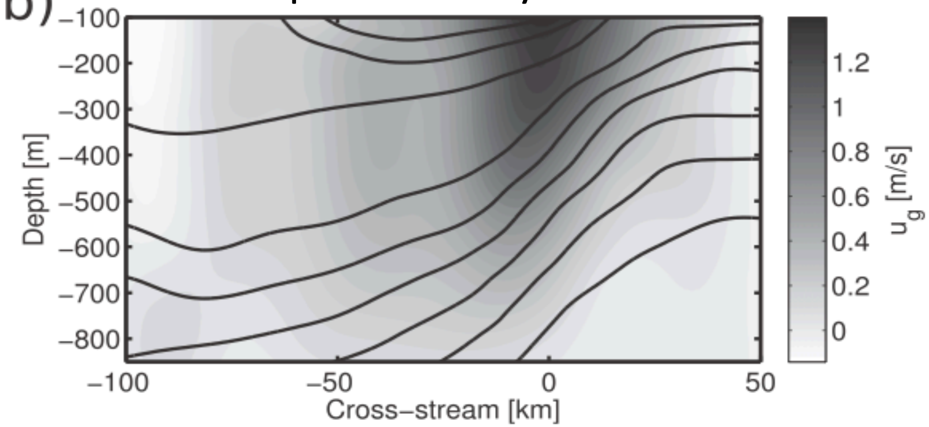


$$Ri_g^{-1} = \left|\partial \mathbf{u}_g / \partial z\right|^2 \Big/ N^2$$

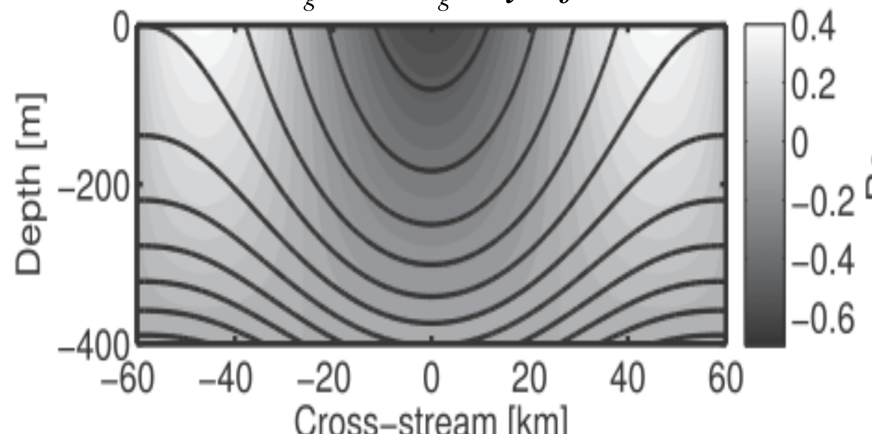


Geostrophic Velocity in Gulf Stream

b)



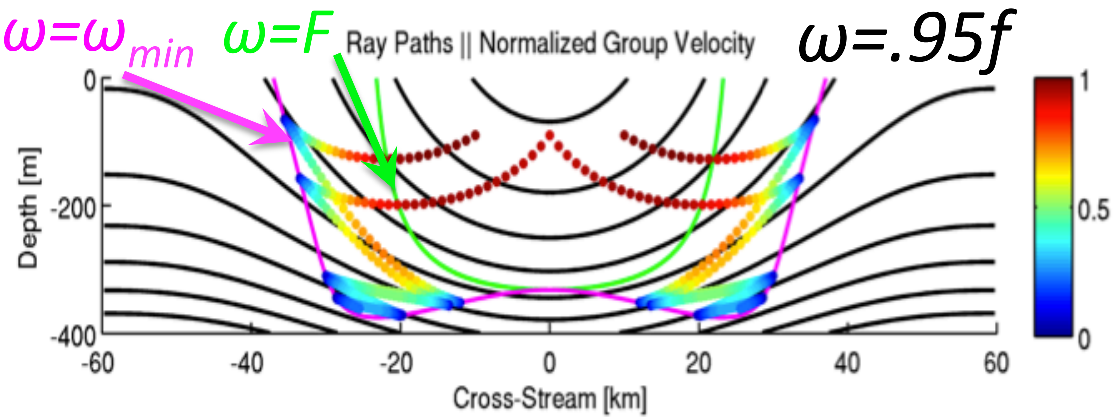
$$Ro_g = -\partial u_g / \partial y / f$$



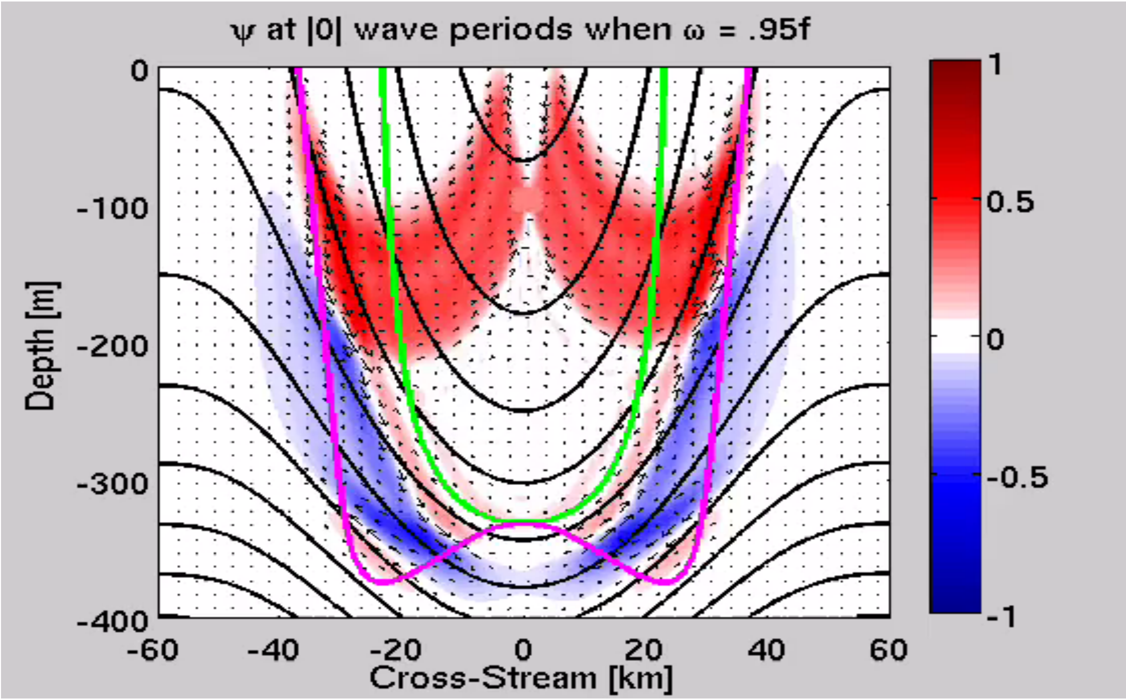
Energy propagates along characteristics

Characteristic slopes are parallel to ray paths

$$\tan(\theta_b) \pm \sqrt{\frac{\omega^2 - \omega_{\text{min}}^2}{N^2}}$$



Numerical Solution for point source forcing



$$\mathcal{L} = (F^2 D_{zz} + 2S^2 D_{zy} + N^2 D_{yy})$$

$$\mathcal{L}\Psi = (\omega - i\mathcal{F})^2 D_{zz}\Psi + \mathbf{b}$$

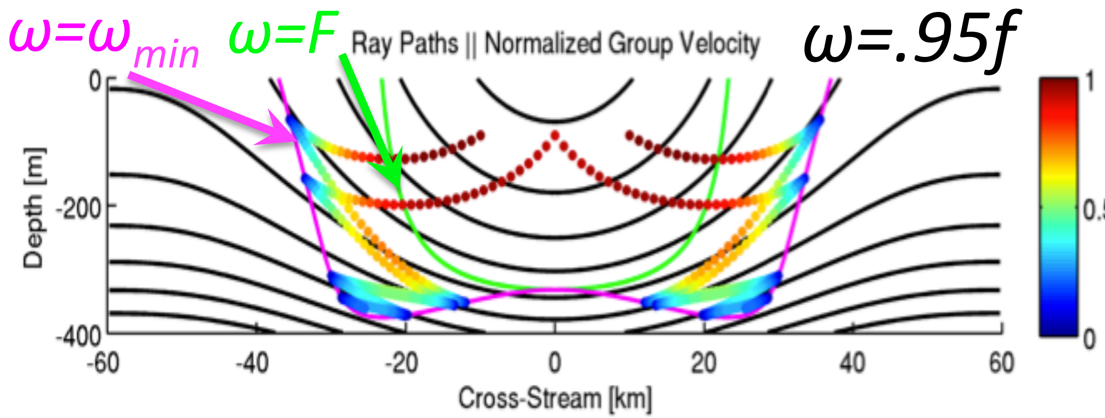
Damping

Forcing at a point with frequency *ω=.95f*

Rays are parallel to characteristics

Characteristic slopes are parallel to ray paths

$$\tan(\theta_b) \pm \sqrt{\frac{\omega^2 - \omega_{\min}^2}{N^2}}$$



<u>The same dispersion relation</u>

$$\omega = \sqrt{F^2 + 2S^2\alpha + N^2\alpha^2},$$
$$\alpha = l / m$$

<u>Plane Wave Physics</u>

$$A = A_0\, e^{i(ly + mz - \omega t)}$$

<u>Group Velocity</u>

$$\mathbf{c}_g = \nabla_{(l,m)}\omega = \frac{(S^2 + N^2\alpha)}{\omega m}(1,\ -\alpha) \approx \frac{N^2(\theta_b - \theta_\delta)}{\omega m}(1, \theta_\delta).$$

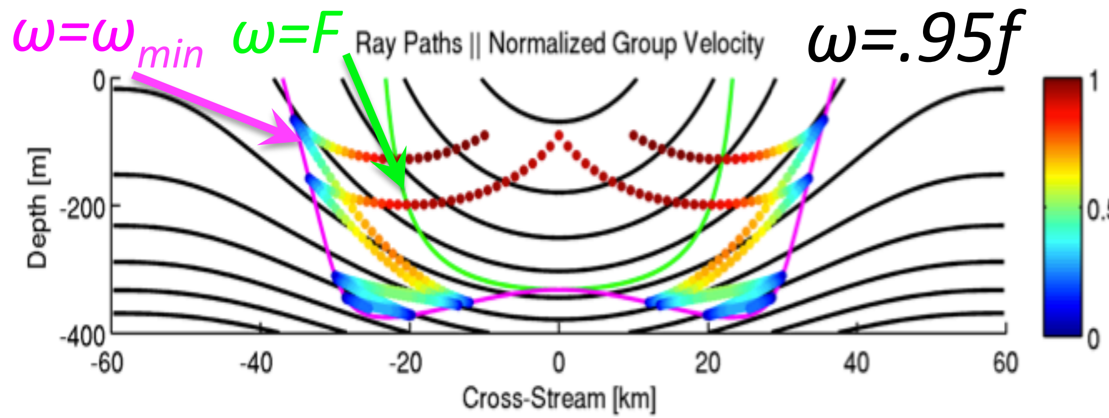
Phase avg. conservation law for wave activity propagation

$$\boxed{\frac{\partial\langle E\rangle}{\partial t} + \nabla\cdot(\mathbf{c}_g\langle E\rangle) = 0}$$

Sub-inertial waves trapped and amplified in a spatially-variable geostrophic flow

Characteristic slopes are parallel to ray paths

$$\tan(\theta_b) \pm \sqrt{\frac{\omega^2 - \omega_{\min}^2}{N^2}}$$



As ω -> $\omega_{\min}$

- Ray slopes approach local isopycnal slope
- Group Velocity -> 0
- Wave energy density increases

Sub-inertial waves trapped and amplified in a spatially-variable geostrophic flow

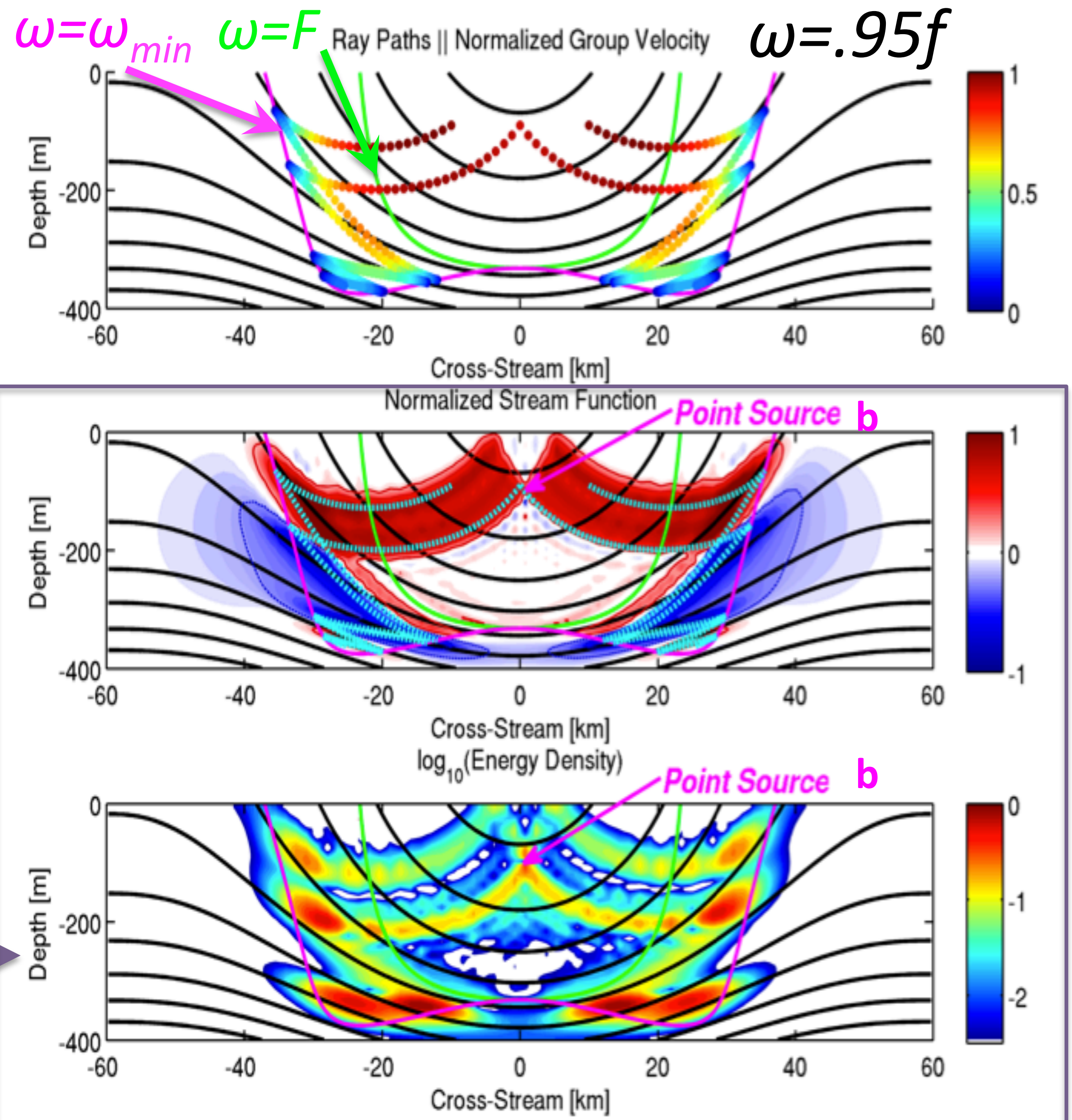
Characteristic slopes are parallel to ray paths

$$\tan(\theta_b) \pm \sqrt{\frac{\omega^2 - \omega_{\min}^2}{N^2}}$$

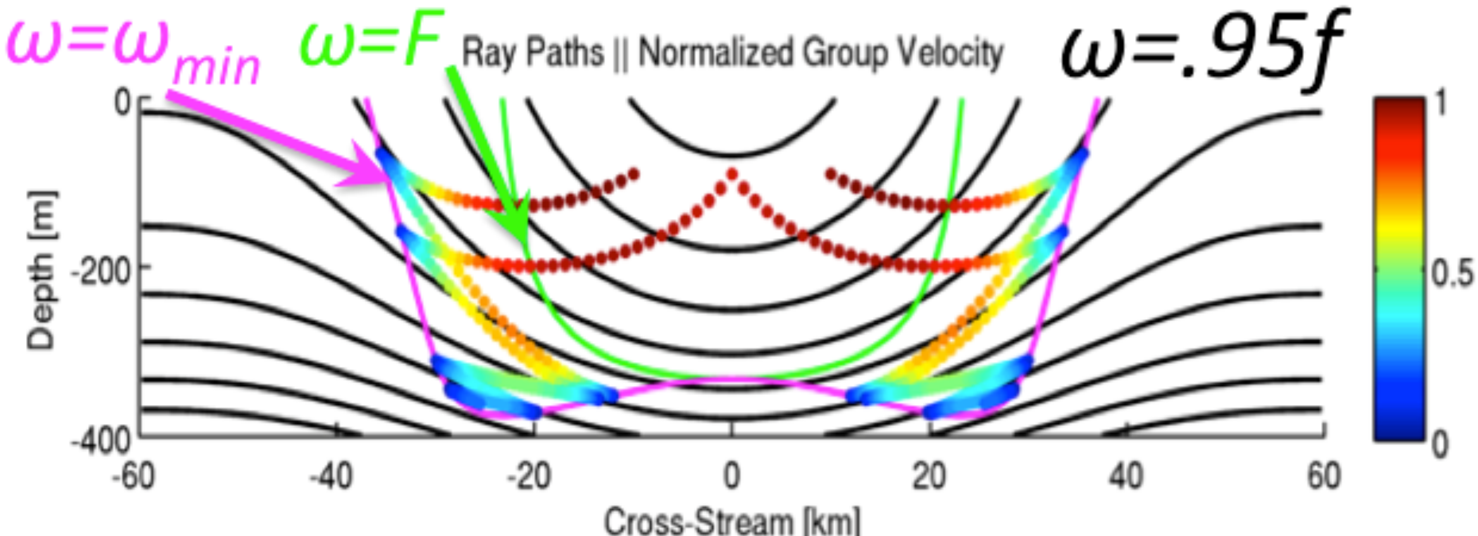
As ω -> ω_{min}

- Ray slopes approach local isopycnal slope
- Group Velocity -> 0
- Wave energy density increases

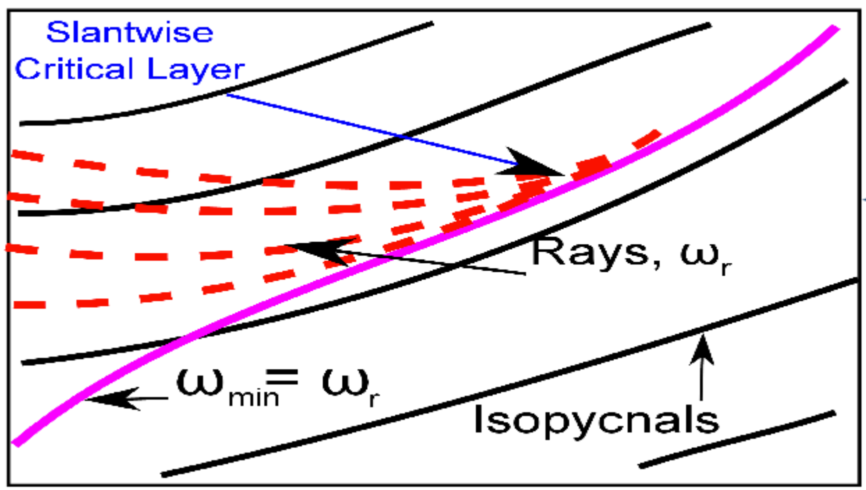
Numerical solution compares well with ray tracing.



Modified Critical Layers



(Activity) x (group velocity) x (ray tube area) = constant **[Lighthill 1978]**.

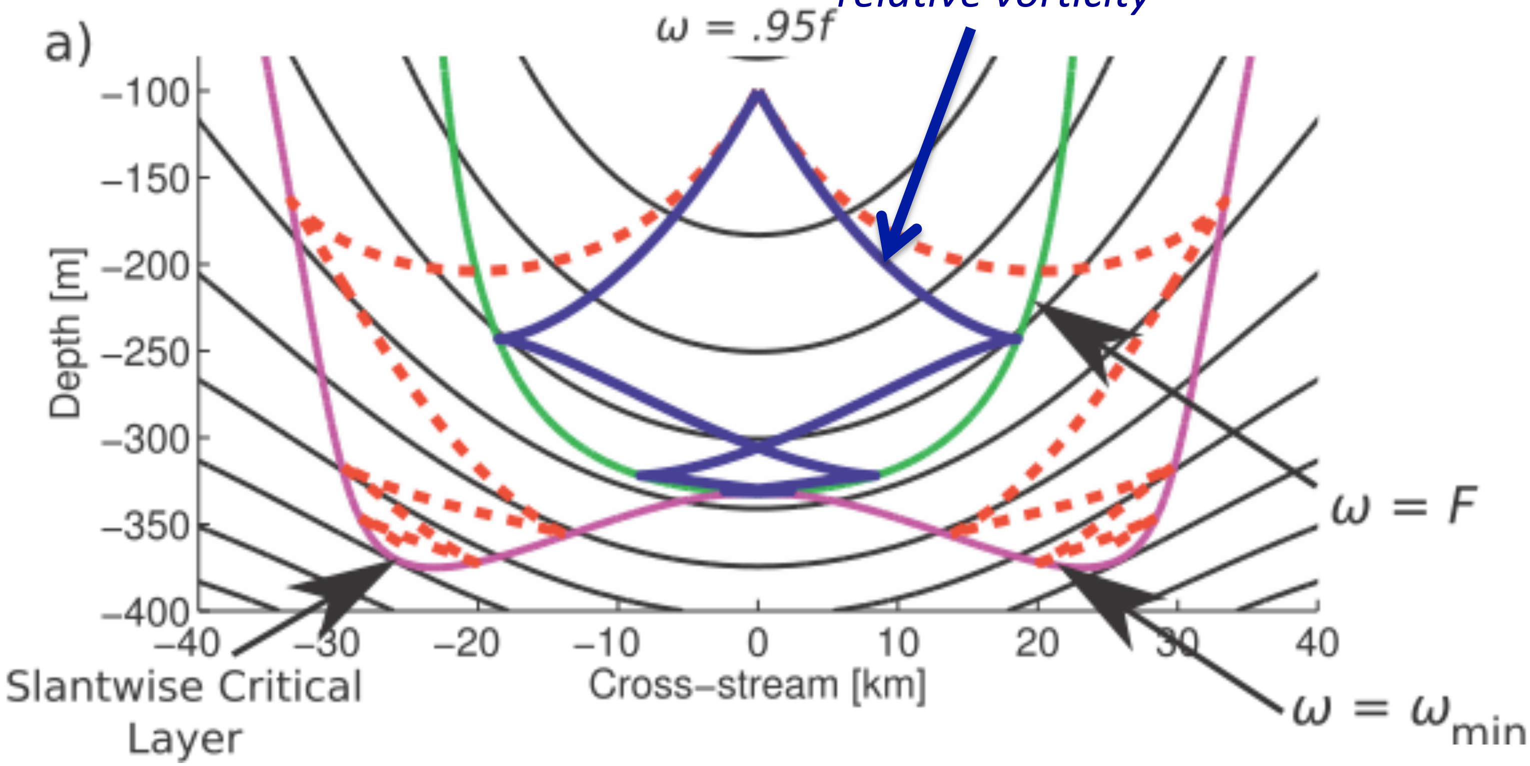


Critical layers occur where isopycnals parallel ω_{min} surfaces

Rays converge to the same line or point and ray tube areas shrink to zero.

Baroclinicity $\omega \approx \sqrt{F^2 \boxed{- 2S^2\theta} + N^2\theta^2}$

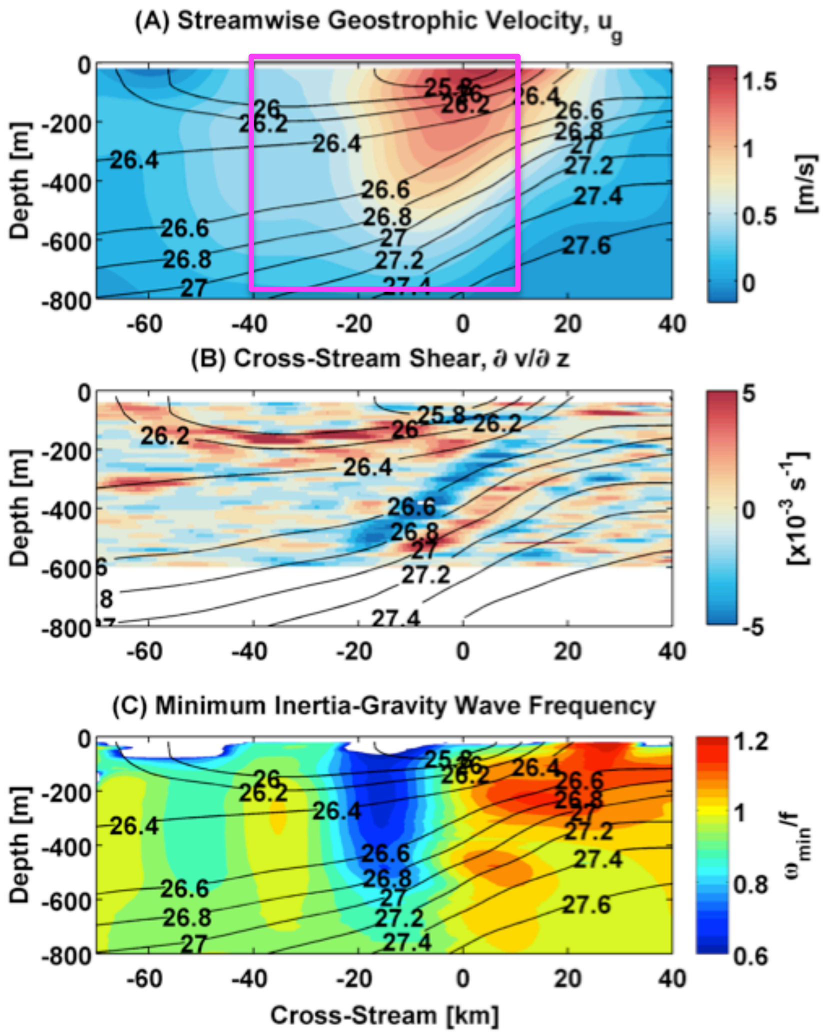
1. lowers the minimum frequency
2. extends the region where sub-inertial waves can exist
3. modifies the geometry of the critical layers



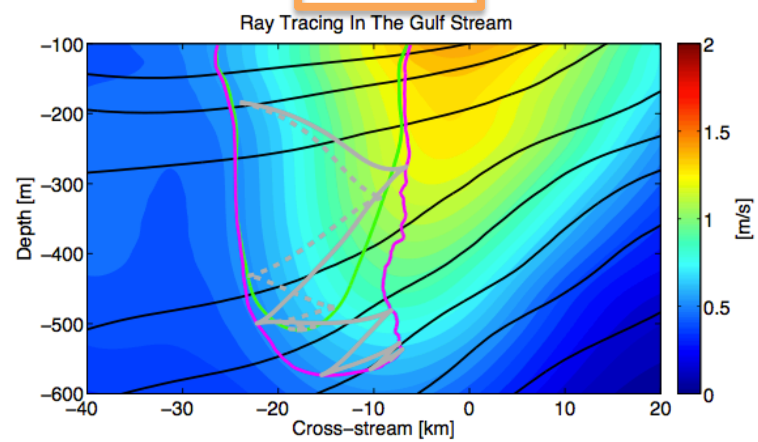
$F^2 = f(f - \partial u_g/\partial y),\ S^2 = f\partial u_g/\partial z = -\partial b_g/\partial y,\ \ \omega_{\min} = \sqrt{F^2 - S^4/N^2} = \sqrt{\frac{fq}{N^2}} = f\sqrt{1 + Ro_g - Ri_g^{-1}}$

Ray tracing predicts trapped/amplified NIW parallel to isopycnals in the Gulf Stream

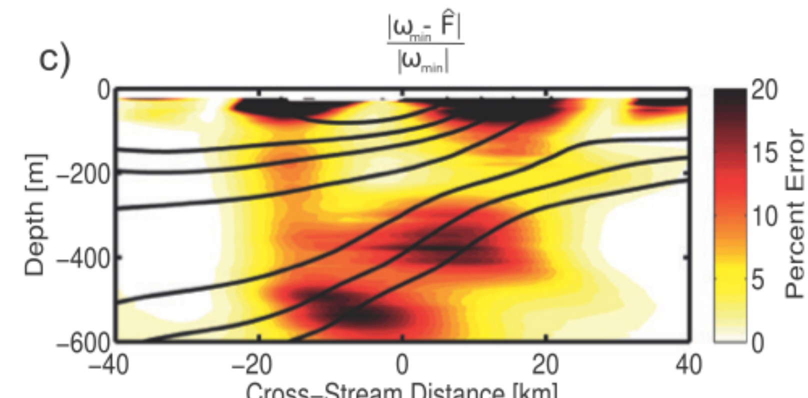
Feb. 2007 observations



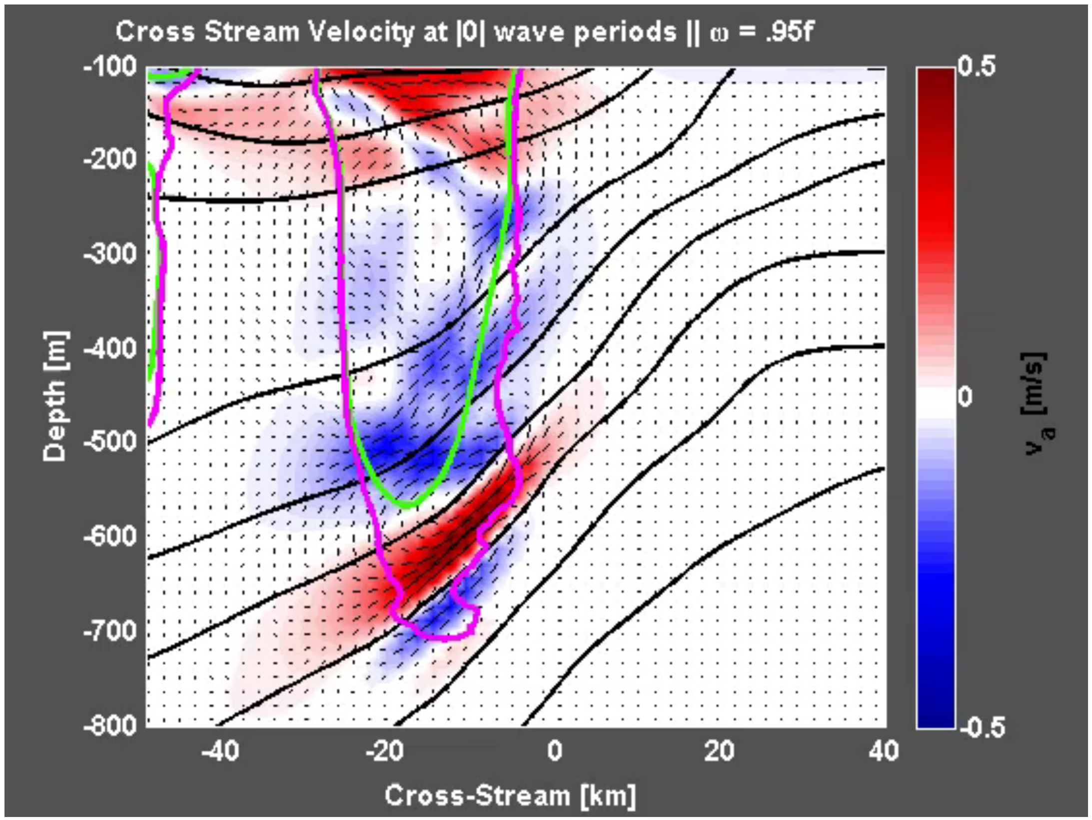
$$\omega \approx \sqrt{F^2 \boxed{-2S^2\theta} + N^2\theta^2}$$



Baroclinic effects modify the dispersion relation by ~ 20 % here



Equilibrium numerical solution of linear equations at constant frequency consistent with ray tracing

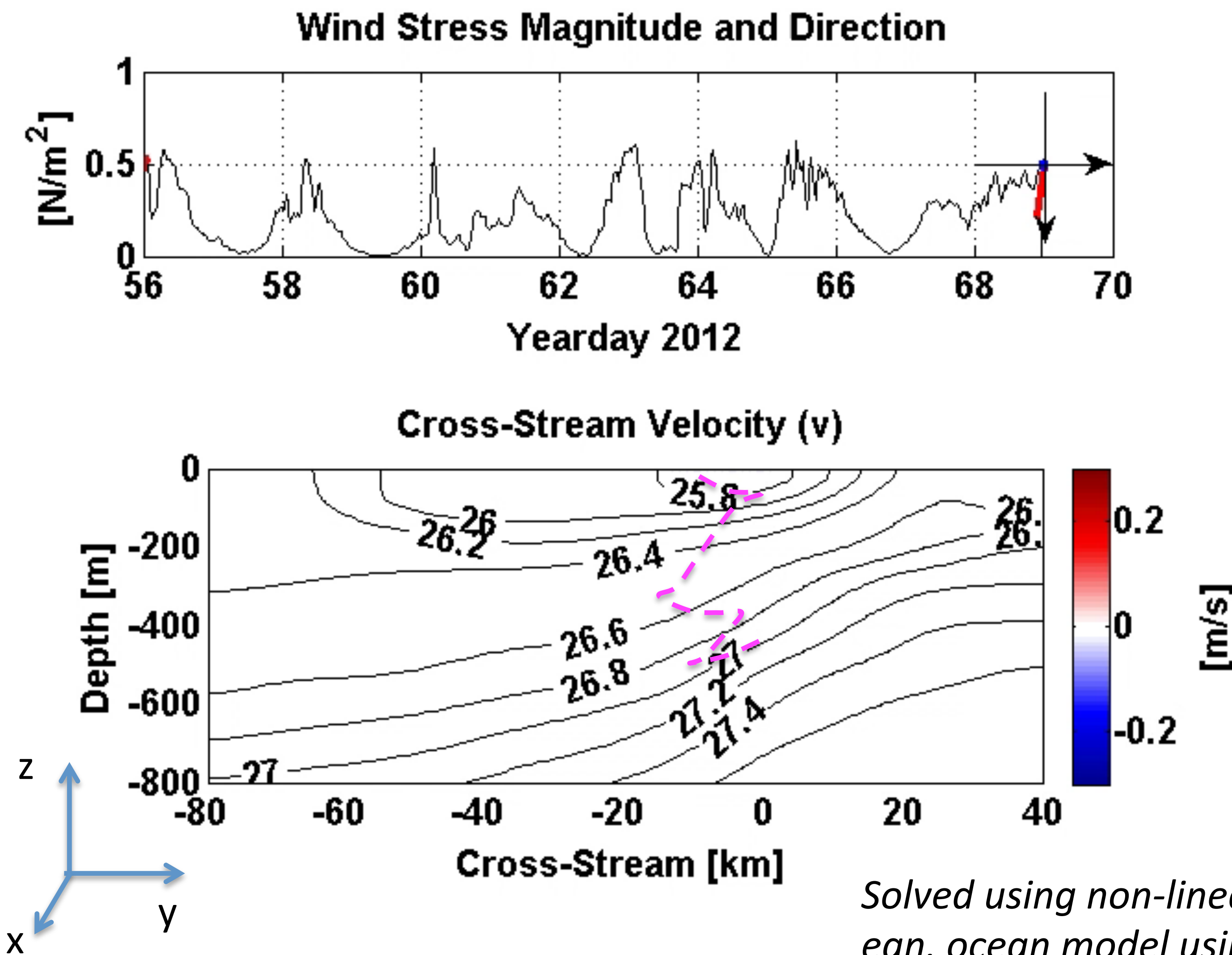


$$\mathcal{L} = (F^2 D_{zz} + 2S^2 D_{zy} + N^2 D_{yy})$$

$$\mathcal{L}\Psi = (\omega - i\mathcal{F})^2 D_{zz}\Psi + \mathbf{b}$$

Can be solved in MATLAB (backslash) in a couple seconds on my laptop

Transient non-linear simulation forced by realistic winds consistent with ray tracing



Solved using non-linear primitive eqn. ocean model using 8 processors for several hrs.

Conclusions

- Sub-inertial waves are trapped and amplified as they approach their minimum frequency:

$$\omega_{\min} = \sqrt{F^2 - S^4/N^2} = \sqrt{\frac{fq}{N^2}} = f\sqrt{1 + Ro_g - Ri_g^{-1}}$$

- Phase lines of minimum frequency oscillations are slanted along isopycnals and the polarization of horizontal velocity is not necessarily circular.

- Ray tracing and numerical solutions illustrate the trapping and amplification of NIWs in regions of strong baroclinicity, similar to observations of banded shear in the observations.