Mathematical Models of Running

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Wednesday September 24, 2008

UMS Talk

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The Pioneer: A.V. Hill

Nobel Prize, 1922.

Hill received the 1922 Nobel Prize for Physiology or Medicine "for his discovery relating to the production of heat in muscle."

Picture found in: Bassett Jr., David R. Scientific Contributions of A.V. Hill: exercise physiology pioneer. *J Appl Physiology.* 93: 1567-1582, 2002.

Why investigate athletics?

- The complaint has been made to me "Why investigate athletics, why not study the processes of industry or of disease?" The answer is twofold. (1) The processes of athletics are **simple and measurable** and carried out to a constant degree, namely to the utmost of a man's powers: those of industry are not; and (2) athletes themselves, being in a state of health and dynamic equilibrium, can be **experimented on without danger** and can repeat their performances exactly again and again.

- I might perhaps state a third reason and say, as I said in Philadelphia, that the study of athletes and athletics is "**amusing**": certainly to us and sometimes I hope to them. Which leads to a fourth reason, perhaps the most important of all: that being "amusing" it may help to bring new and enthusiastic recruits to the study of physiology, which needs every one of them, especially if they be chemists.

Quotations found in: Bassett Jr., David R. Scientific Contributions of A.V. Hill: exercise physiology pioneer. *J Appl Physiology.* 93: 1567-1582, 2002.

Paraphrased

- Athletics is simple, measurable
- Athletes are good experimental subjects
- It's cool!

A.V. Hill Model

- Based on Newton's Laws and Thermodynamics

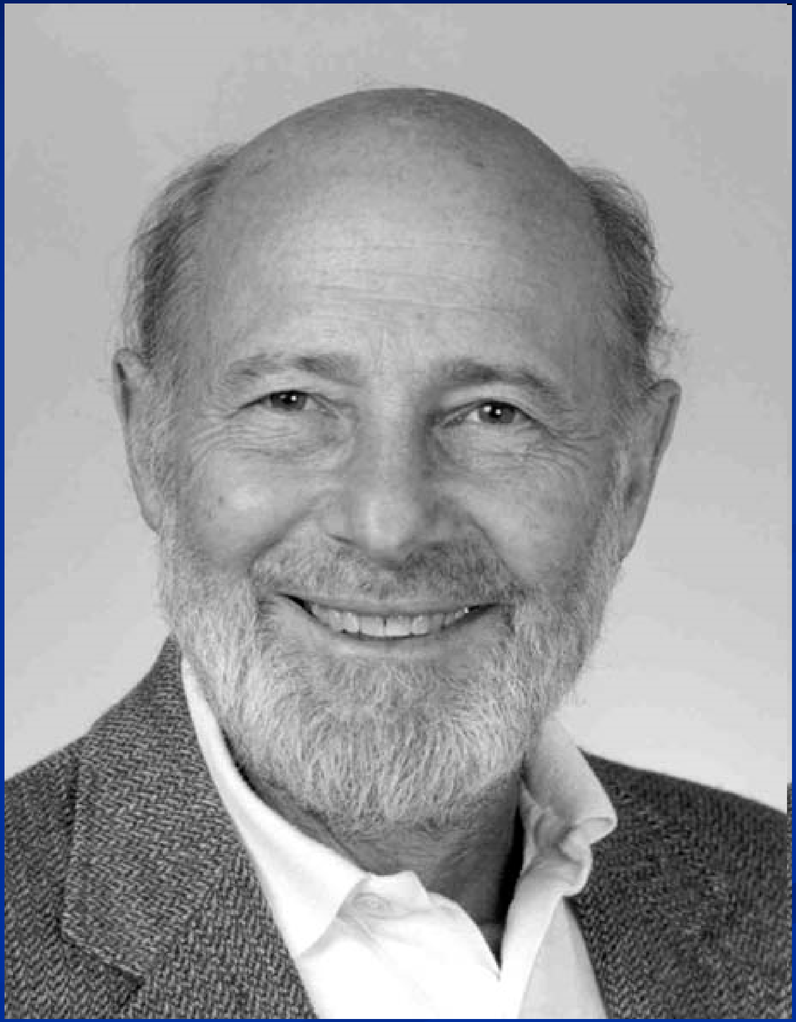
- The ideal Human Engine is 38% efficient

- Not much mathematics. A lot of physiology.

Hill, A.V. The Air-Resistance to a Runner. *Proceedings of the Royal Society of London, Series B, Containing Papers of a Biological Character,* Vol. 102, No. 718, 380-385, 1928.

Joseph B. Keller

- 400+ publications!
- Developed the Geometrical Theory of Diffraction
- Developed the EBK (Einstein-Brillouin-Keller) method of quantization



Biographical info and picture from: http://math.stanford.edu/~keller/

The Keller Model

Built primarily upon the physical foundations of the Hill Model

1. Conservation of Linear Momentum
2. Conservation of Energy
3. Oxygen consumption yields energy
4. Cast as an Optimal Control Problem

Keller, Joseph B. A Theory of Competitive Running, *Physics Today*, **26**, No. 9, 42-47, 1973.

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Keller's Problem

"We wish to determine how a runner should vary his speed $v(t)$ during a race of distance D in order to run it in the shortest time."

Variables

v(t) = velocity

f(t) = propulsive force per unit mass

E(t) = available energy (oxygen) in the muscles per unit mass

T = total time

D = total distance

Physiological Constants

- *F = maximum possible value of f(t)*
- $E_0 = E(0)$*, initial available energy (oxygen)*
- σ *= rate at which energy (oxygen) is supplied*
- τ *= a constant related to the resistive force per unit mass*

Six Defining Equations

1. $$D = \int_0^T v(t)\,dt$$

2. $$\frac{dv}{dt} + \frac{v}{\tau} = f(t)$$

3. $$v(0) = 0$$

4. $$f(t) \le F$$

5. $$\frac{dE}{dt} = \sigma - f(t)v(t)$$

6. $$E(t) \ge 0$$

Mathematical Problem

Given T, and the four physiological constants,

Find the three functions $v(t)$, $f(t)$ and $E(t)$ satisfying the six equations such that

$$D = \int_0^T v(t)\,dt$$ **is maximized.**

Simplification

- Express $f(t)$ and $E(t)$ in terms of $v(t)$ by substitution within the six defining equations.

$$\frac{dv}{dt} + \frac{v}{\tau} = f(t) \quad \textit{and} \quad f(t) \le F \quad \Rightarrow \boxed{\frac{dv}{dt} + \frac{v}{\tau} \le F}$$

Eqn. 2 Eqn. 4 Eqn. 7

$$\frac{dE}{dt} = \sigma - f(t)v(t) \Rightarrow \frac{dE}{dt} = \sigma - v(t)\left(\frac{dv}{dt} + \frac{v(t)}{\tau}\right)$$

Eqn. 5

Then integrate the last expression to obtain:

$$\boxed{E(t) = E_0 + \sigma t - \frac{v^2(t)}{\tau} - \frac{1}{\tau}\int_0^t v^2(s)\,ds \ge 0}$$

Eqn. 8

Reformulated Problem

Maximize $D = \int_0^T v(t)\,dt$ **such that**

1. $v(0) = 0,$

2. $\dfrac{dv}{dt} + \dfrac{v}{\tau} \leq F,$

3. $E_0 + \sigma t - \dfrac{v^2(t)}{\tau} - \dfrac{1}{\tau}\int_0^t v^2(s)\,ds \geq 0$

First: The Solution

Then: Some Computation

Solution: 2 Cases

- The Race is a short sprint or “dash,” with typical parameters, $D \leq 291$ meters.
- The Race is longer distance.

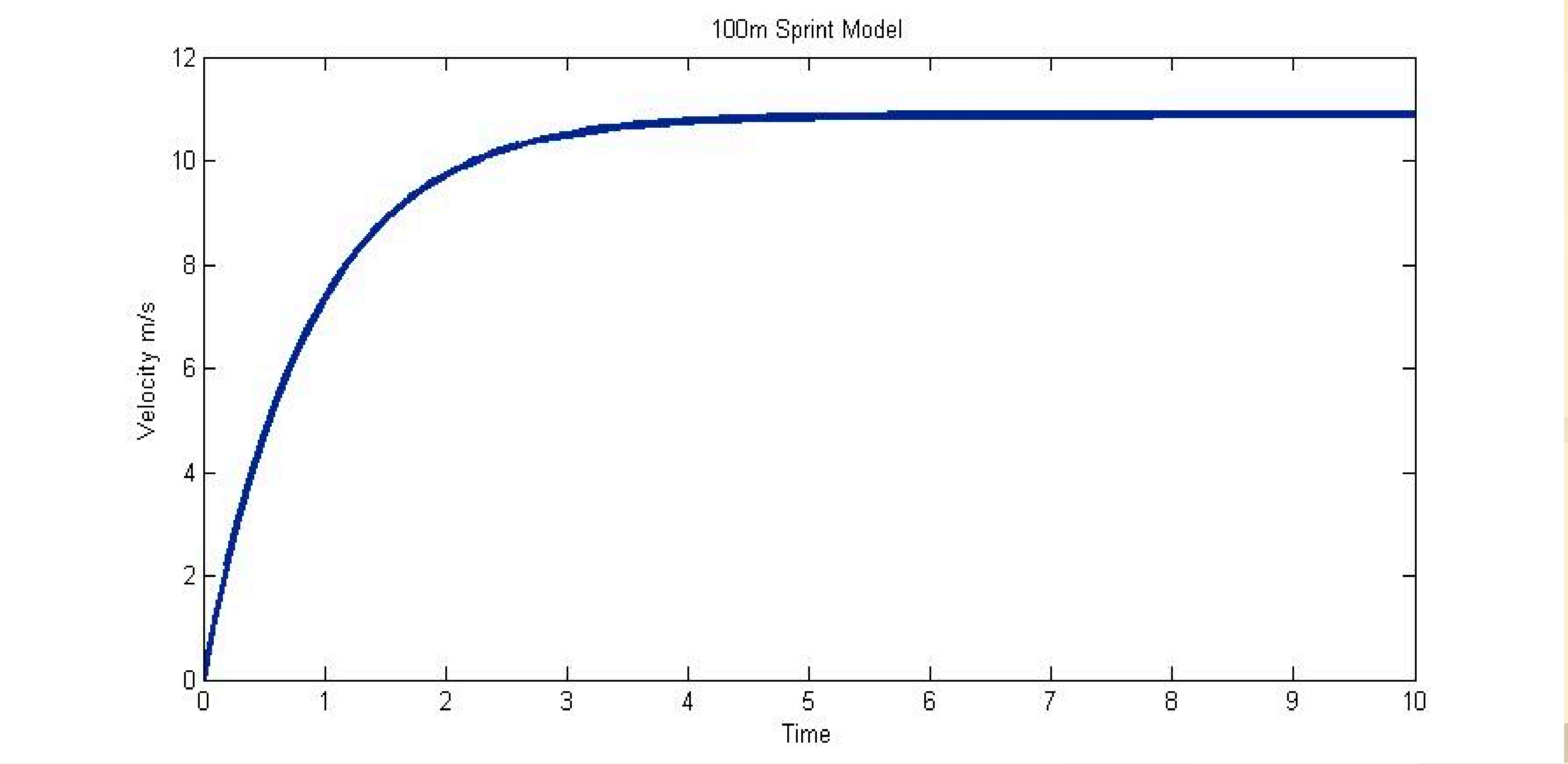
Case 1: The Dash

$$f(t) = F$$ for all t.

$$\boxed{v(t) = F\tau(1 - e^{-t/\tau})}$$

$$\Rightarrow D = F\tau^2\left(\frac{T}{\tau} + e^{-T/\tau} - 1\right) \quad 0 \le T \le T_c$$

Case 1: The Graph of *v*(*t*)



$$v(t) = F\tau(1 - e^{-t/\tau})$$

Case 2: The Longer Distances

$$T \geq T_c \Leftrightarrow D \geq D_c$$

3 Phases

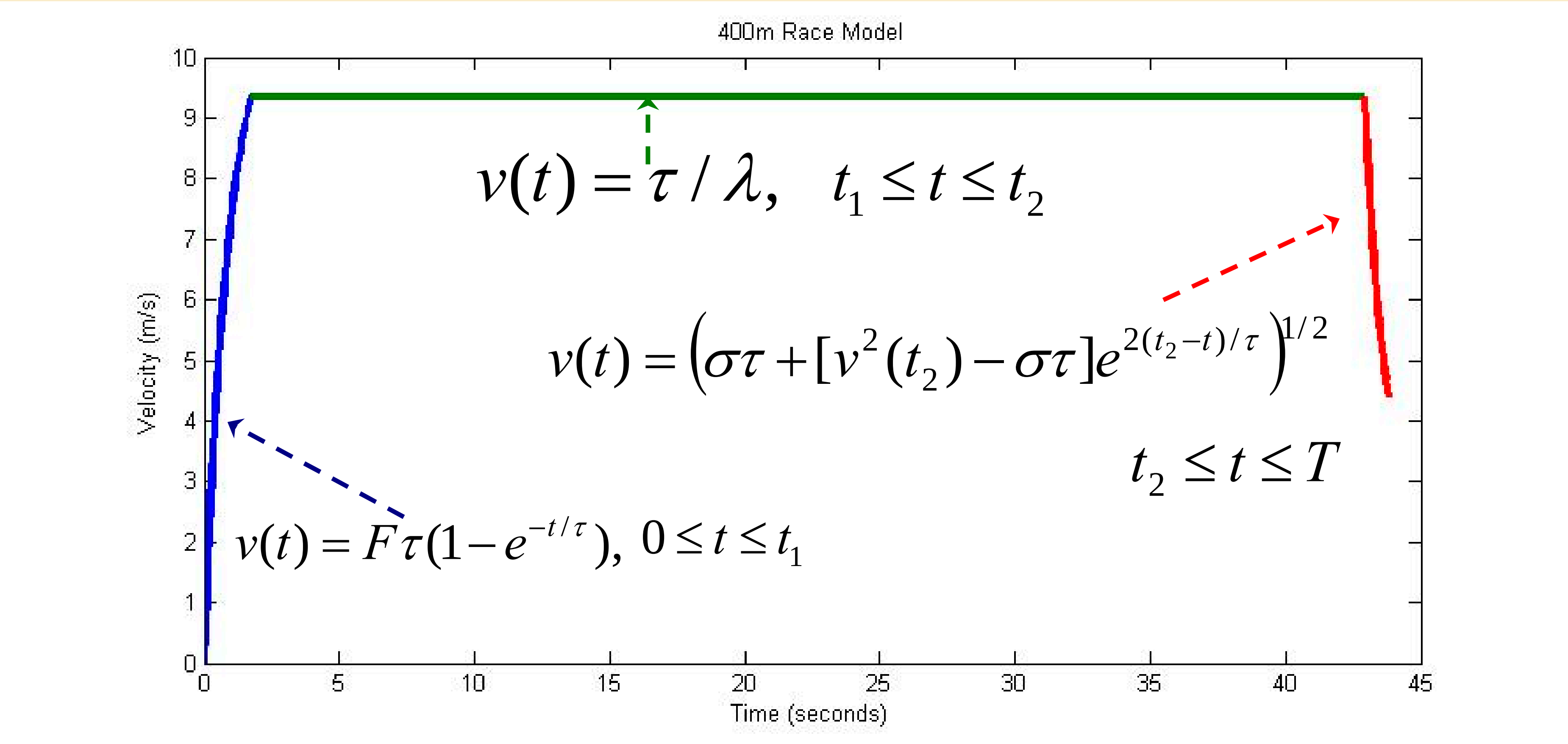
1. **Acceleration:** $v(t) = F\tau(1 - e^{-t/\tau}) \quad 0 \leq t \leq t_1$

2. **Cruising: $v(t)$ = constant** $\quad t_1 \leq t \leq t_2$

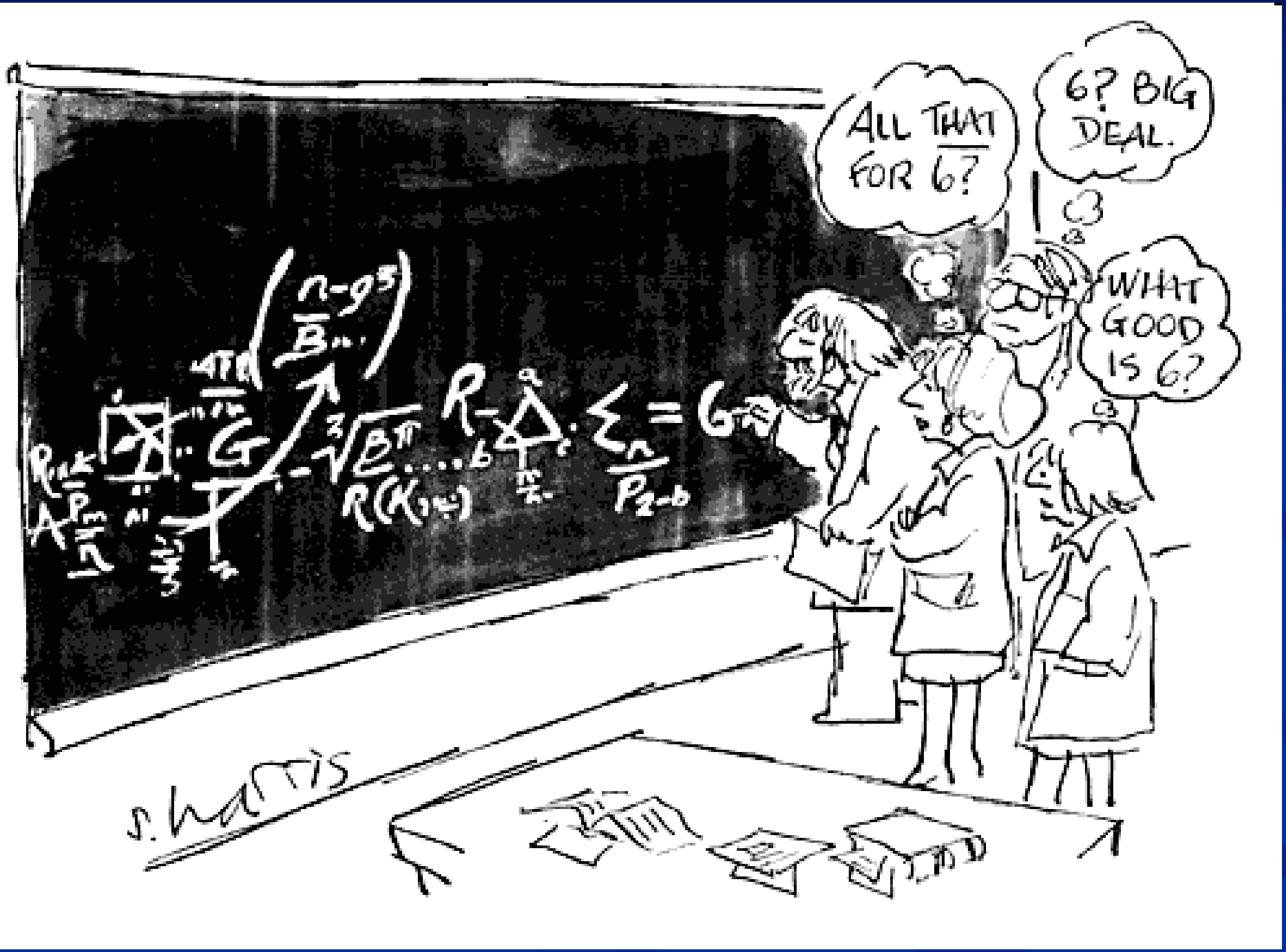
3. **Deceleration:** $\quad t_2 \leq t \leq T$

$$v(t) = \left(\sigma\tau + [v^2(t_2) - \sigma\tau]e^{2(t_2 - t)/\tau}\right)^{1/2}$$

Case 2: The Graph of $v(t)$



The Details…



Comic Found online: http://www.math.kent.edu/~sather/PHOTOS/math03.gif

Calculus of Variations

- The calculus of *functionals*.
- *Functional =* "function of a function"
- The goal is usually associated with finding a maximizing or minimizing function of a given functional.
- See Kirk and Wikipedia

More Precise Definitions

1. Functionals and the Calculus of Variations

Definition 1.1. *A **functional**, in our case, is a function $J : \mathcal{D} \to \mathcal{R}$ where the range $\mathcal{R}$ is the field of real numbers and the domain $\mathcal{D}$ is a space of functions mapping $[0, T]$ to $\mathbb{R}$, each containing 3 arcs (or differentiable curves) on $[0, t_1]$, $[t_1, t_2]$, and $[t_2, T]$.*

Definition 1.2. *The **first variation** of a functional $J(v)$ is defined as follows:*

$$\delta J(v, \delta v) = \frac{d}{d\epsilon} J(v + \epsilon \delta v)|_{\epsilon=0}$$

One Key Step in Case 2

- After we determine an expression for $D > D_c$, we must choose v to maximize D subject to the constraint $E(t_2) = 0$.
- To do this, we consider the functional $D + \lambda E(t_2)/2$ where λ is a Lagrange multiplier.
- We then compute the first variation of this functional and set it equal to zero.

Computation…

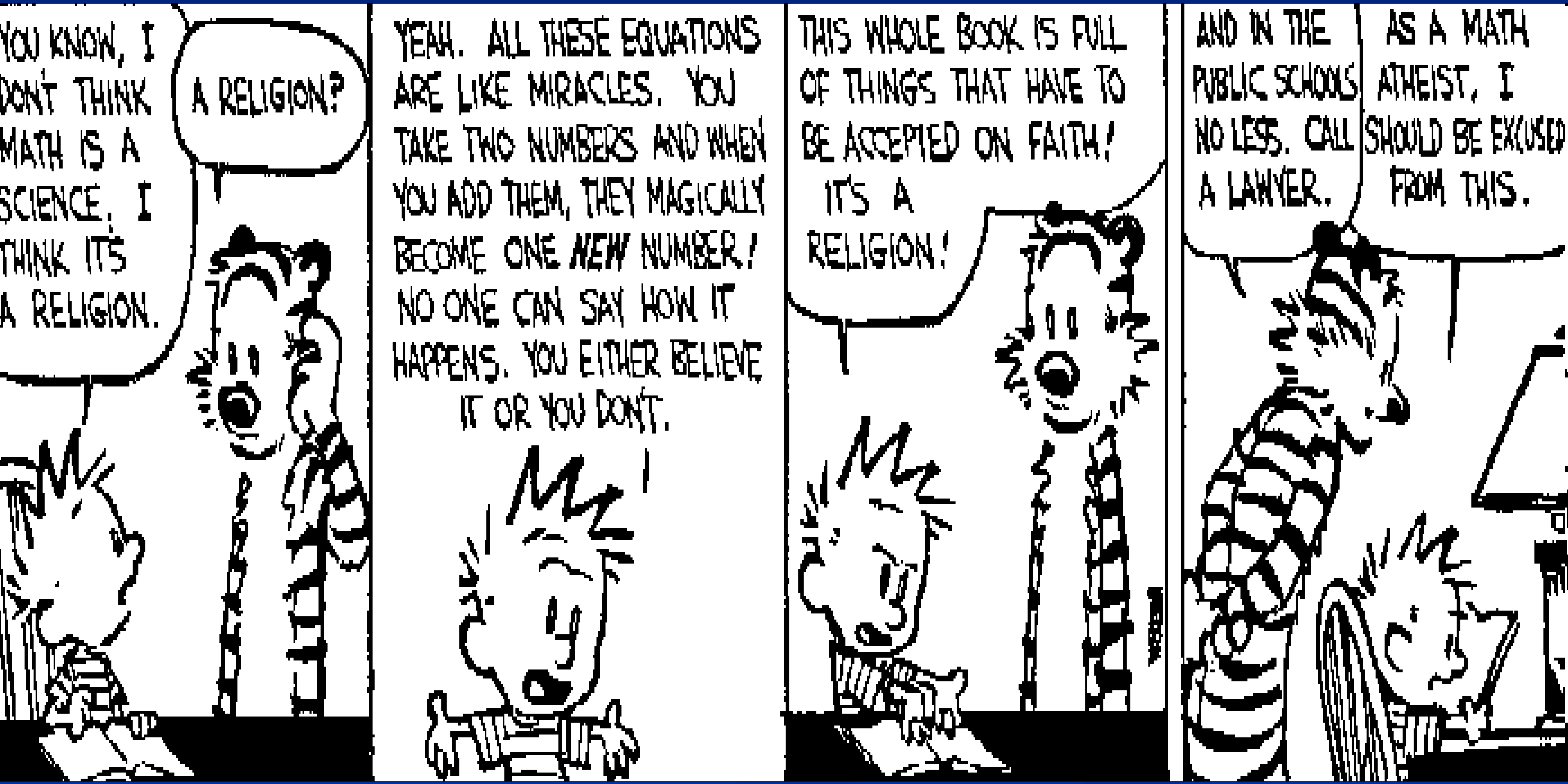


Image found at: http://home.uchicago.edu/~slapan/Links/images/Math.gif

OK, you can wake up now.

What's Missing?

- Accurate for races 400m and less.
- What really happens in the races longer than 400m?
- Even in "solo" world record attempts the races are run with a "negative split."
- There is typically an acceleration in the later part of the race, either quickly and forcefully to defeat competitors, or gradually over the final third of the race to expend all remaining energy more evenly.

How can we revise the model?

- Add variables to improve the accuracy of the fatigue factors.
- Use a 3 compartment hydraulic model to more accurately represent our knowledge of physiology – see Behncke
- Is there another way?

Uncertainty

- When $D > D_c$, there exists some uncertainty about what constant rate to attempt between time t_1 and t_2.
- The athlete does not know the values of E_0 and σ and only learns once the race has begun based upon feelings.

New Problem, New Solution

- Problem: Determine the best pacing strategy given your objective.
- Put a probability distribution on σ and E_0.
- Then choosing Keller's pace will result in a 50% failure rate.
- Perhaps you want a greater chance of finishing before running out of energy, be conservative.
- Or perhaps you want to risk it, go out hard, test your limits.