Mathematical Models for Optimal Running Performance

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1. A Very Quick Primer Example

Compute the first variation of $J(y) = \int yy'dx$.

Use the definition of the first variation.

2. Computation of formula (3.9) in Keller (1974)

We consider the functional $J = D + \lambda E(t_2)/2$ as indicated by Keller. $E(t)$ is given by 2.2 and D is given by 3.8. Hence we have the following expression for J:

$$J(v) = \int_0^{t_1} F\tau(1-e^{-t/\tau})dt + \int_{t_1}^{t_2} v(t)dt + \int_{t_2}^{T} \left\{\sigma\tau + [v^2(t_2) - \sigma\tau]e^{2(t_2-t)/\tau}\right\}^{1/2} dt$$

$$+\frac{\lambda}{2}\left[E_0 + \sigma t_2 - \frac{v^2(t_2)}{2} - \frac{1}{\tau}\int_{t_1}^{t_2} v^2(s)ds - \frac{1}{\tau}\int_0^{t_1} F^2\tau^2\left(1-e^{-t/\tau}\right)^2 ds\right]$$

Then we consider the first variation of J which is given by the definition:

$$\delta J \equiv \frac{d}{d\epsilon}(J(v+\epsilon h))|_{\epsilon=0}$$

$$\Rightarrow \delta J = \frac{d}{d\epsilon}|_{\epsilon=0}\int_0^{t_1} F\tau(1-e^{-t/\tau})dt + \frac{d}{d\epsilon}|_{\epsilon=0}\int_{t_1}^{t_2} v(t)dt$$

$$+\frac{d}{d\epsilon}\left(\epsilon\int_{t_1}^{t_2} h(t)dt + \int_{t_2}^{T}\left\{\sigma\tau + [v^2(t_2) + 2\epsilon v(t_2)h(t_2) + \epsilon^2h^2(t_2) - \sigma\tau]e^{2(t_2-t)/\tau}\right\}^{1/2} dt\right)|_{\epsilon=0}$$

$$+\frac{d}{d\epsilon}|_{\epsilon=0}\left(\frac{\lambda}{2}\left[E_0 + \sigma t_2 - \frac{v^2(t_2) + 2\epsilon v(t_2)h(t_2) + \epsilon^2h^2(t_2)}{2}\right]\right)$$

$$-\frac{d}{d\epsilon}|_{\epsilon=0}\left(\frac{\lambda}{2\tau}\int_{t_1}^{t_2} v^2(s) + 2\epsilon v(s)h(s) + \epsilon^2h^2(s)ds + \frac{\lambda}{2\tau}\int_0^{t_1} F^2\tau^2\left(1-e^{-t/\tau}\right)^2 ds\right)$$

Then we differentiate with respect to epsilon. We observe that the first two terms and last term are constant in epsilon and hence are zero. We can also differentiate under the integral sign since epsilon is not the variable of integration:

$$\Rightarrow \delta J = \int_{t_1}^{t_2} h(t)dt$$

$$+\frac{1}{2}\int_{t_2}^{T} e^{2(t_2-t)/\tau}(2v(t_2)h(t_2)+2\epsilon h^2(t_2))\left\{\sigma\tau + [v^2(t_2) + 2\epsilon v(t_2)h(t_2) - \epsilon^2h^2(t_2) - \sigma\tau]e^{2(t_2-t)/\tau}\right\}^{-1/2} dt$$

$$+\frac{\lambda}{2}\left[-v(t_2)h(t_2) - \epsilon h^2(t_2) - \frac{1}{\tau}\int_{t_1}^{t_2} 2v(s)h(s) + 2\epsilon h^2(s)ds\right]|_{\epsilon=0}$$

Now we evaluate at $\epsilon = 0$:

$$\Rightarrow \delta J = \int_{t_1}^{t_2} h(t)dt + \frac{1}{2}\int_{t_2}^{T} e^{2(t_2-t)/\tau}(2v(t_2)h(t_2))\left\{\sigma\tau + [v^2(t_2)-\sigma\tau]e^{2(t_2-t)/\tau}\right\}^{-1/2} dt$$

$$+\frac{\lambda}{2}\left[-v(t_2)h(t_2) - \frac{1}{\tau}\int_{t_1}^{t_2} 2v(s)h(s)ds\right]$$

Now we set δJ equal to zero and obtain the following eqn :

$$\int_{t_1}^{t_2} h(t)dt + \frac{1}{2}\int_{t_2}^{T} e^{2(t_2-t)/\tau}(2v(t_2)h(t_2))\left\{\sigma\tau + [v^2(t_2)-\sigma\tau]e^{2(t_2-t)/\tau}\right\}^{-1/2} dt$$

$$+\frac{\lambda}{2}\left[-v(t_2)h(t_2) - \frac{1}{\tau}\int_{t_1}^{t_2} 2v(s)h(s)ds\right] = 0$$

Now we seek a $v(t)$ such that this expression holds for arbitrary $h(t) \in C_0^\infty(\Omega)$ where $C_0^\infty(\Omega)$ is the space of all infinitely differentiable functions defined on Ω (an open subset of $\mathbb{R}$) whose support is a compact set contained in Ω. We observe that the coefficients of $h(t_2)$ and $h(t)$ must vanish so we will get two equations.

$$v(t) = \tau/\lambda \quad \text{for} \quad t_1 \le t \le t_2$$

$$\lambda = 2\int_{t_2}^{T} e^{2(t_2-t)/\tau}\left\{\sigma\tau + [v^2(t_2)-\sigma\tau]e^{2(t_2-t)/\tau}\right\}^{-1/2} dt$$

We have now obtained the main result: v is a constant. In our final computation we are using the fundamental lemma of calculus of variations or more generally the **DuBois-Reymond Lemma**:

Suppose that f is **locally integrable** and defined on an open set $\Omega \subset \mathbb{R}^n$. If

$$\int_\Omega f(x)h(x)dx = 0$$

for all $h \in C_0^\infty(\Omega)$, then $f(x) = 0$ for almost all $x \in \Omega$ ($x \neq 0$ possibly on a set of measure 0 contained in Ω). Here, $C_0^\infty(\Omega)$ is the space of all infinitely differentiable functions defined on Ω whose support is a compact set contained in Ω.

Note: Let f be a Lebesgue measureable fundtion on an open set $\Omega \in \mathbb{R}^n$ where $f: \Omega \to \mathbb{R}$ if for all compact subsets $K \subset \Omega$:

$$\int_K |f|d\mu < \infty$$

Example: $e^{|x|}$ is locally integrable but not globally integrable.

See Stein and Shakarchi's Princeton Lectures in Analysis: Real Analysis for a good but intense book on advanced real analysis (p. 105 for a definition of locally integrable) or Wikipedia.