A brief introduction to oceanic waves

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Thank you for the invitation to speak, visit, and even more, **thank you for your hospitality!**

It's an honor to be here.

My goal for this talk

A descriptive introduction to observations and mathematical models of oceanic wave motions.

My aspirations for you

After this talk, I hope that you will have an appreciation for:

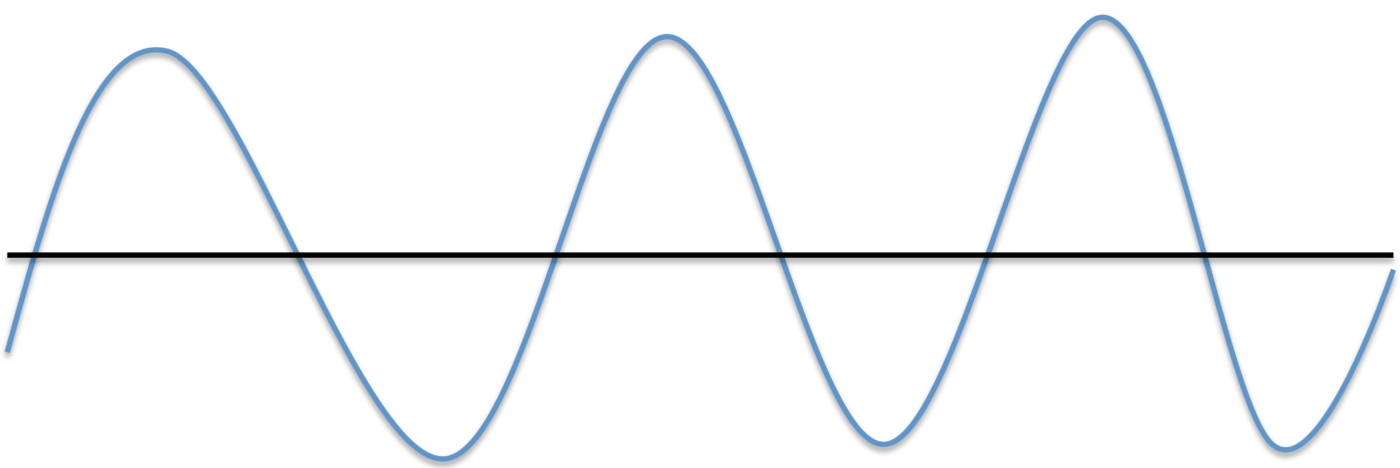
1. Ubiquity and diversity of ocean waves
2. Mathematical models that have been applied to study them

What this talk will **not** be

- A particularly **original** or **comprehensive** treatment of ocean wave motion.
- Ocean waves have been a subject of constant and often intense research for much of the last several centuries, at least back to 1776 when Laplace studied the tides.
- These researchers (and teachers) deserve credit for building up all the ideas presented today.

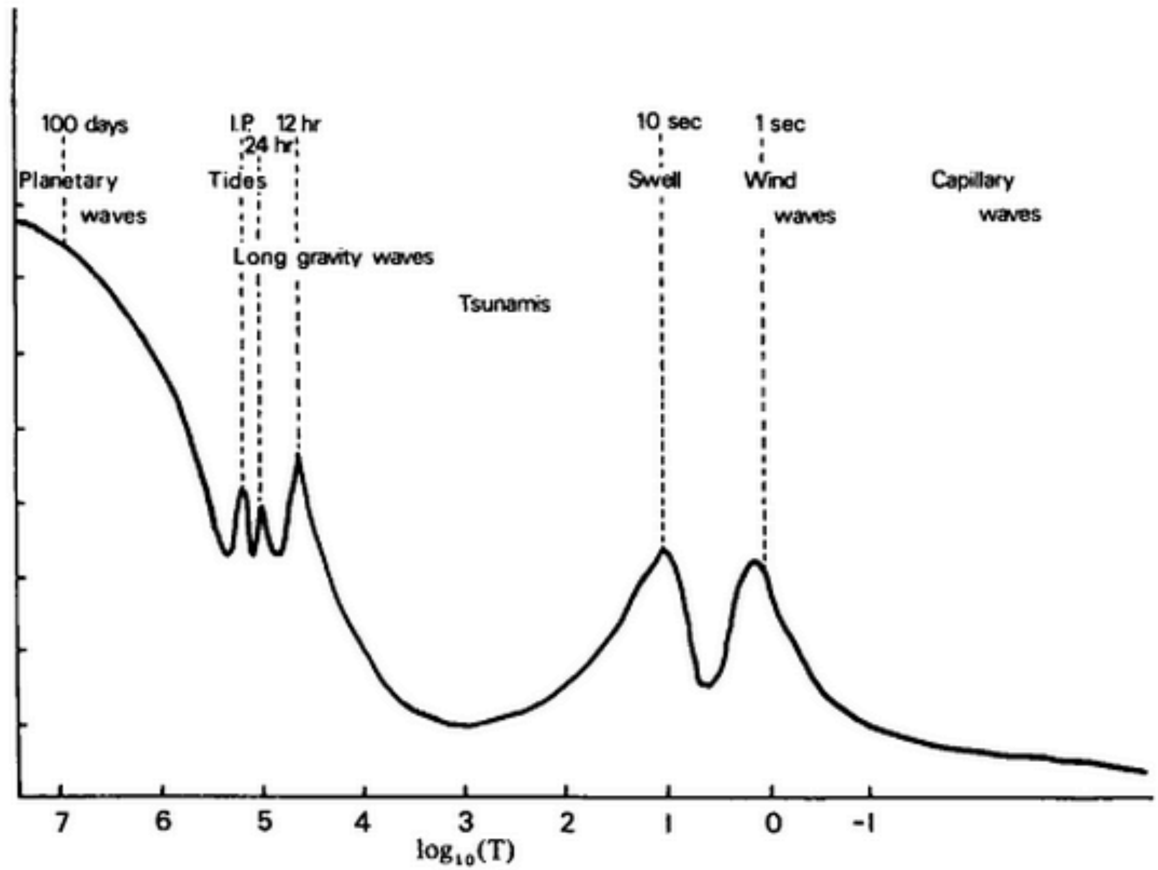
What is a wave?

- Loosely defined
- A transient/oscillatory perturbation to a background state that travels at a speed typically different from the speed of the background.



Ocean is characterized by many frequencies

- Many classes of wave motion.
- Classified by dominant restoring force
- Spectral peaks associated with a particular energy source/resonance (e.g. semi-diurnal tides)



Log$_{10}$(time scale)

Schematic frequency spectrum from Leblond and Mysak 1978

Navier Stokes System: One equation to rule them all

$$\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\text{Time tendency}} + \underbrace{\mathbf{u}\cdot\nabla\mathbf{u}}_{\text{Advection}} + \underbrace{\mathbf{f}\times\mathbf{u}}_{\text{Coriolis}} = -\underbrace{\frac{\nabla p}{\rho_0}}_{\text{Pressure Gradient}} + \underbrace{\mathbf{b}}_{\text{Buoyancy}} + \underbrace{\nu\Delta\mathbf{u}}_{\text{Friction}}$$

$$\frac{\partial b}{\partial t} + \mathbf{u}\cdot\nabla b = \kappa\Delta b,$$

$$\nabla\cdot\mathbf{u} = 0,$$

$$\mathbf{f} = (0, 0, 2\Omega_e \sin(\theta)) \text{ where } \Omega_e \approx 7.3\times 10^{-5}\ \mathrm{s}^{-1}$$

$$\mathbf{f}\times\mathbf{u} = (-fv, fu, 0)$$

Traditional approimation

$$f = f_0 + \beta y$$

Beta-plane approximation

$$b = -g\rho/\rho_0$$

buoyancy

-> Generality is both the greatest asset and greatest weakness of the Navier Stokes system.

-> Need to create reduced models to understand particular problems and obtain or interpret solutions.

Tangent-plane Cartesian coordinates

2π/f$_0$ ~ 1 day in midlatitudes

Local tangent plane approximation (beta plane)

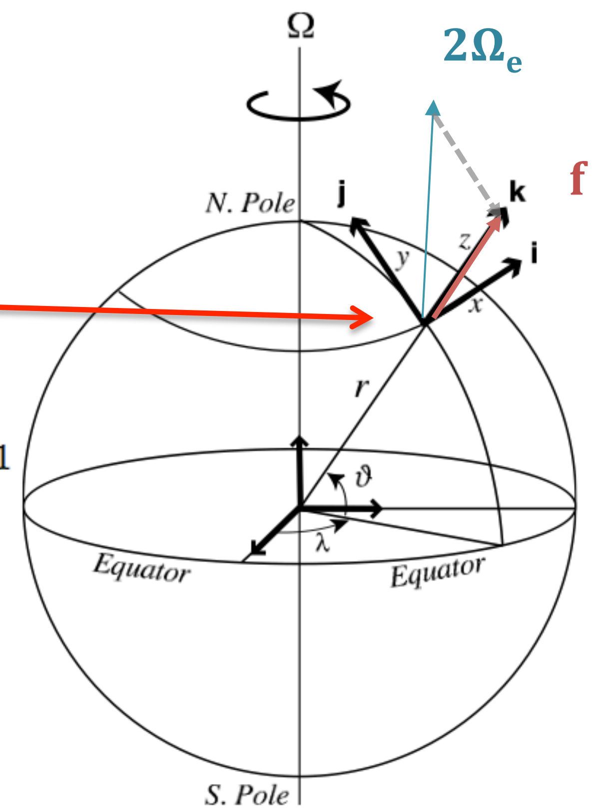


Fig. 2.3 The spherical coordinate system. The orthogonal unit vectors **i**, **j** and **k** point in the direction of increasing longitude λ, latitude ϑ, and altitude z. Locally, one may apply a Cartesian system with variables x, y and z measuring distances along **i**, **j** and **k**.

$$\mathbf{f} = (0, 0, 2\Omega_e \sin(\theta)) \text{ where } \Omega_e \approx 7.3 \times 10^{-5}\ \mathrm{s}^{-1}$$

$$\mathbf{f} \times \mathbf{u} = (-fv, fu, 0)$$

$$f = f_0 + \beta y$$

$$\text{unit vectors } (\mathbf{i}, \mathbf{j}, \mathbf{k})$$

Outline: A tale of three waves

1. Surface-gravity waves
2. Internal inertia-gravity waves
3. Rossby waves

Three different reduced models of the Navier-Stokes system.

Example 1:
Gravity waves at the air-water interface

- Familiar to the casual observer
- Generated by local (or distant) winds.
- Underwater bathymetry guides waves onto beaches and leads to steepening and breaking as they approach shore

Assumptions/scaling

Dimensional Parameters

$\tilde{U}$	Velocity scale [m/s]
$\tilde{L} = 2\pi/\tilde{k}$	Length scale [m] / Wavenumber [rad/m]
$\tilde{T} = 2\pi/\tilde{\omega}$	Time scale [s] / Frequency [rad/s]

Explicit Assumptions

Coriolis force is negligible

$$\frac{f}{\tilde{\omega}} << 1$$

Advection is negligible

$$\frac{\tilde{U}\tilde{k}}{\tilde{\omega}} << 1$$

Frictional force is negligible

$$\frac{\nu\tilde{k}^2}{\tilde{\omega}} << 1$$

Implicit Assumptions

1. All components of velocity scale the same
2. Horizontal and vertical length scales are the same

$$\tilde{U} \sim \tilde{V} \sim \tilde{W}$$

$$\tilde{L} \sim \tilde{H}.$$

Assumptions/scaling

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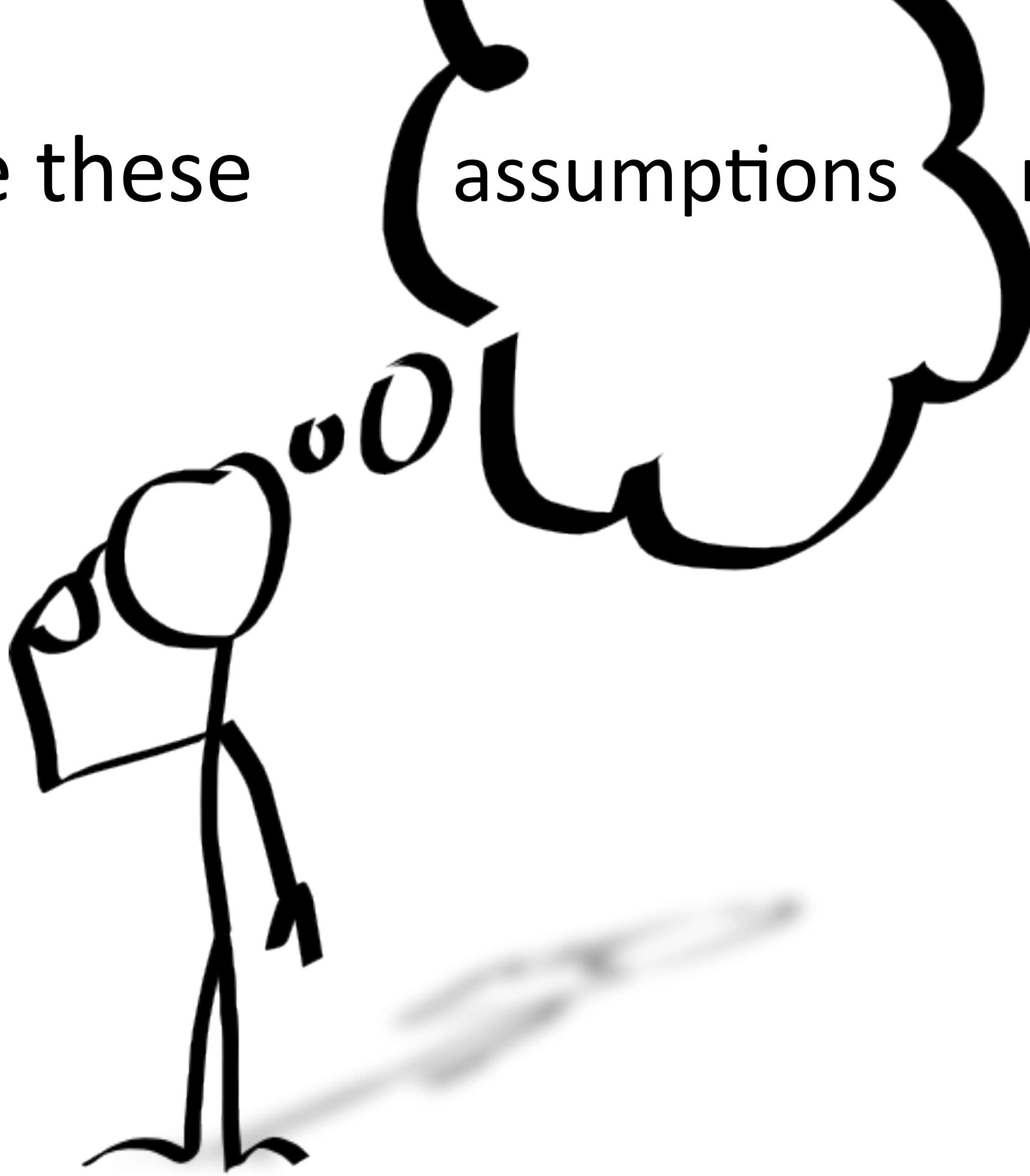
Ratios of two timescales

Are these assumptions reasonable?



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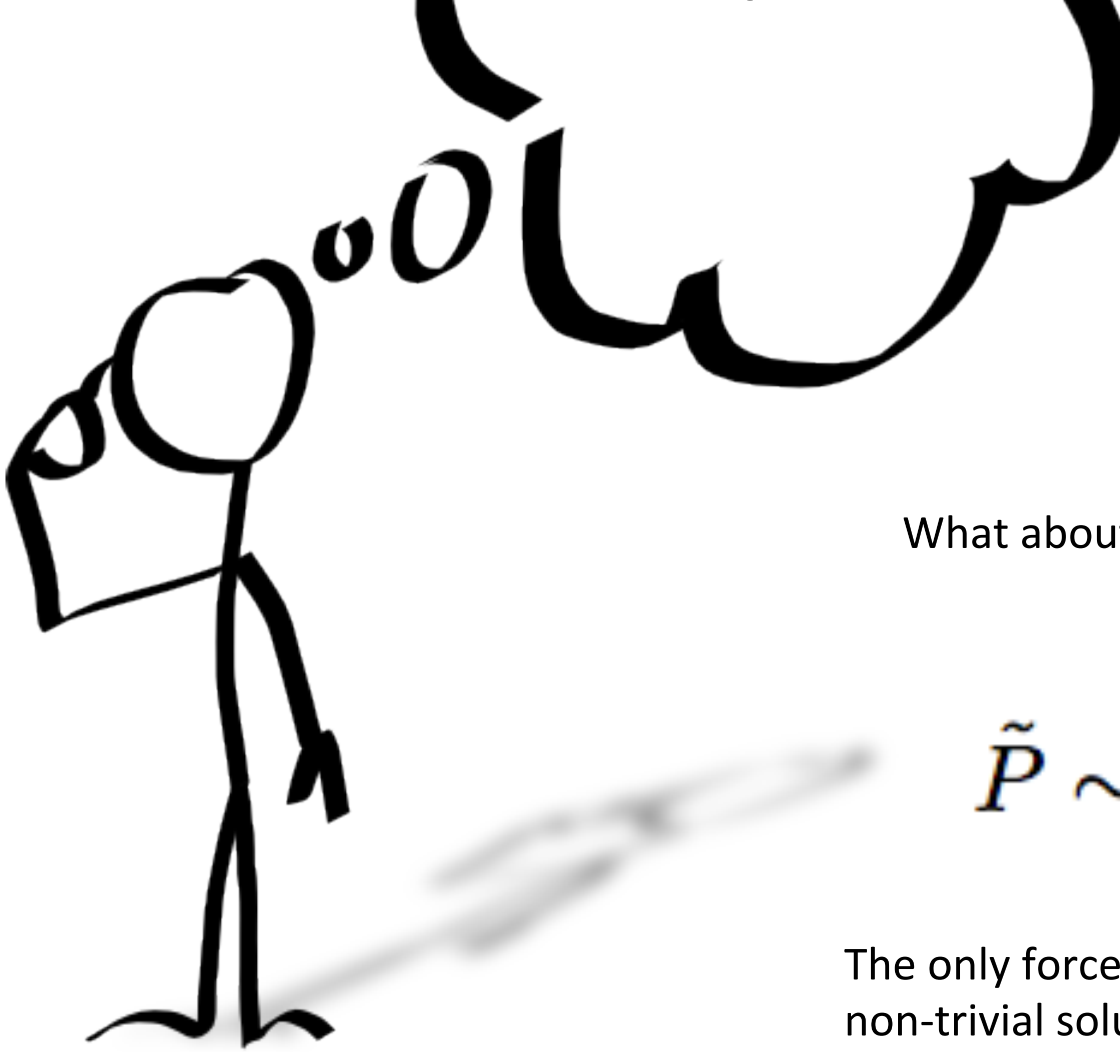


What about pressure?

$$\tilde{P} \sim \tilde{U}\tilde{\omega}/\tilde{k}$$

The only force left to produce a non-trivial solution!

Reduced governing equations

$$\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\text{Time tendency}} = - \underbrace{\frac{\nabla p}{\rho_0}}_{\text{Pressure Gradient}} + \underbrace{\mathbf{g}}_{\text{Buoyancy}}$$

$$\nabla \cdot \mathbf{u} = 0,$$

b = g = (0,0,g) – the buoyancy is effectively constant

Note: scaling assumptions imply: $\omega >> N = \sqrt{\frac{\partial b}{\partial z}} = \sqrt{\frac{-g}{\rho_0}\frac{\partial \rho}{\partial z}}$

Irrotational flow!

$$\frac{\partial \nabla \times \mathbf{u}}{\partial t} = 0$$

Helmholtz Theorem: any three-dimensional vector field on a compact domain can be decomposed into a rotational and divergent part

$$\mathbf{u} = \underbrace{\nabla \phi}_{\text{irrotational/divergent}} + \underbrace{\nabla \times \mathbf{\Phi}}_{\text{solenoidal/rotational}}$$

$$\text{where } \nabla \cdot \mathbf{\Phi} = 0.$$

Therefore: governing equation reduces to a **harmonic function** because both the **curl** and **divergence** of **u** are zero:

$$0 = \nabla \cdot \mathbf{u} = \Delta \phi$$

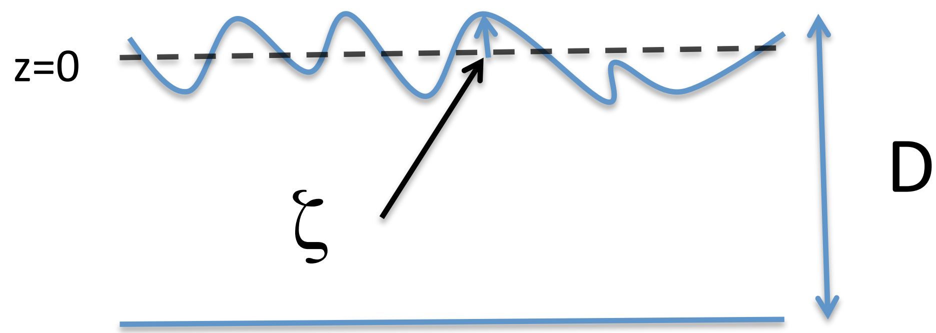
Boundary Conditions

"Many people would claim that the boundary conditions are not part of physics but belong to metaphysics or religion….Yet all the evidence is that it evolves in a regular way according to certain laws." "The Quantum State of the Universe", Nuclear Physics (1984)

Boundary Conditions

Bottom: w = 0

$$\frac{\partial \phi}{\partial z} = 0, \text{ at } \; z = -D$$



Surface:

1) kinematic

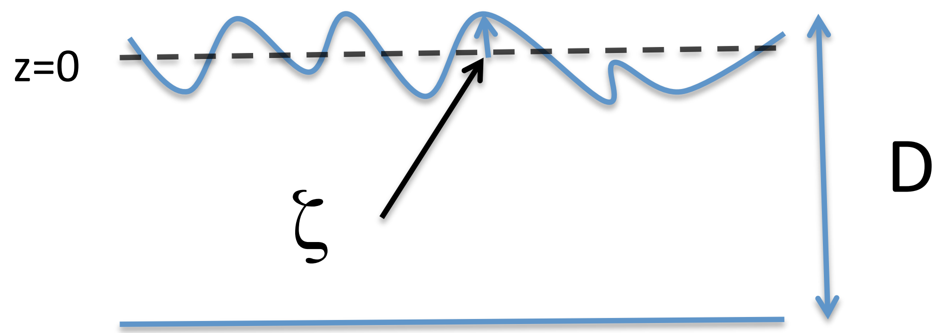
$$w|_{z=\zeta} = \frac{\partial \phi}{\partial z}|_{z=\zeta} = \frac{\partial \zeta}{\partial t} + \nabla_h \phi \cdot \nabla_h \zeta, \quad \text{where } \nabla_h = (\partial/\partial x, \partial/\partial y)$$

2) Dynamic (zero pressure difference across the surface)

Boundary Conditions

Bottom: w = 0

$$\frac{\partial \phi}{\partial z} = 0, \text{ at } \ z = -D$$



Taylor series expansion shows that we must apply the surface BCs at z=0 (not zeta) to be asymptotically consistent

-> Combined surface BC at z = 0:

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0$$

Solution

Insert ansatz: $$\phi = Re(R(z)e^{i(kx+ly-\omega t)})$$

Solve ODE for vertical structure: $$\frac{d^2R}{dz^2} = K^2R \qquad K^2 = k^2 + l^2$$

$$R = A' \cosh(K(z+D))$$

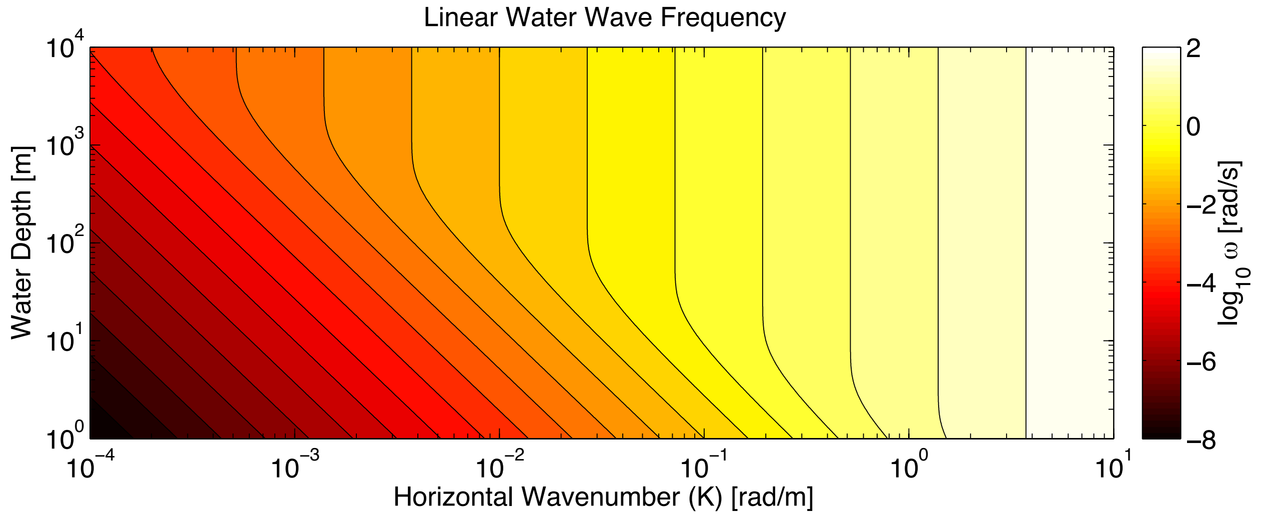
Surface BC => $$-\omega^2 \cosh(KD) + gK \sinh(KD) = 0$$

$$\omega = \pm\sqrt{gK \tanh(KD)}$$

General solution is a sum of Fourier components.

Dispersion Relation

- Frequency is a function of depth and wavenumber.



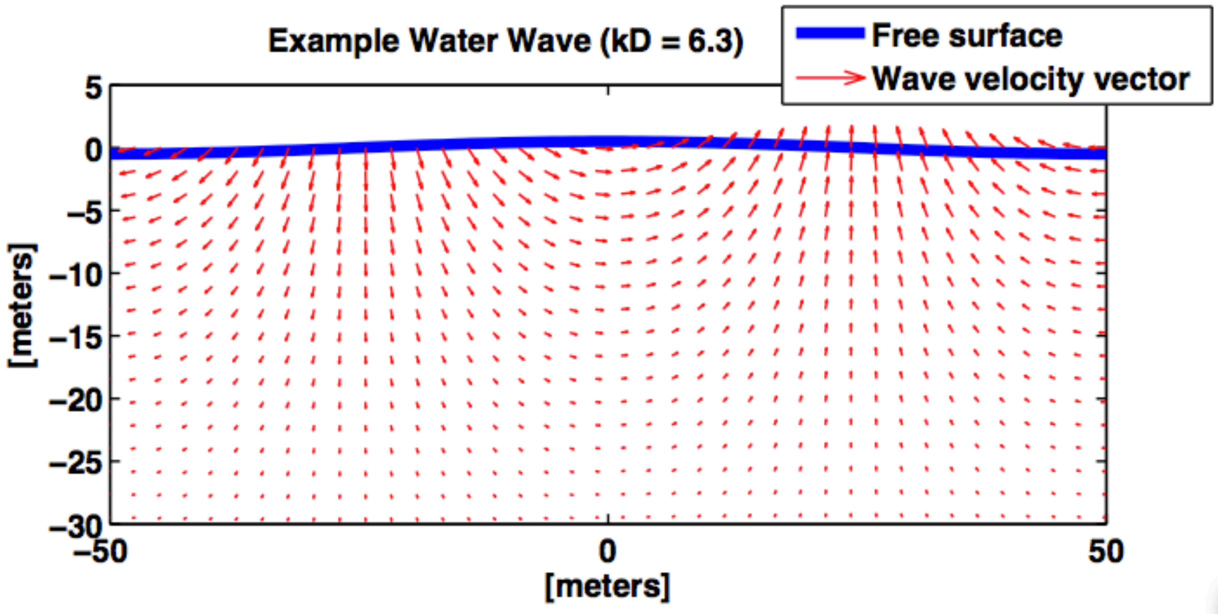
Dispersive waves: *in contrast to the solutions of the classic wave equation, which travel without changing shape, the different Fourier components of these water-wave solutions will become separated in space, wavenumber, and frequency as they propagate.*

What do these solutions look like?

- Use *polarization relations* assuming $K = k$ and $l = 0$

$$
\begin{aligned}
\zeta(x,t) &= \zeta_0 \cos(kx - \omega t), \\
\phi(x,z,t) &= \zeta_0 \frac{\omega}{k} \frac{\cosh k(z+D)}{\sinh KD} \sin(kx - \omega t), \\
u &= \zeta_0 \omega \cos(kx - \omega t) \frac{\cosh k(z+D)}{\sinh KD}, \\
w &= \zeta_0 \omega \sin(kx - \omega t) \frac{\sinh k(z+D)}{\sinh KD}.
\end{aligned}
$$

What do these solutions look like?



Three regimes:

High KD (short / deep waves)

Low KD (shallow / long waves)

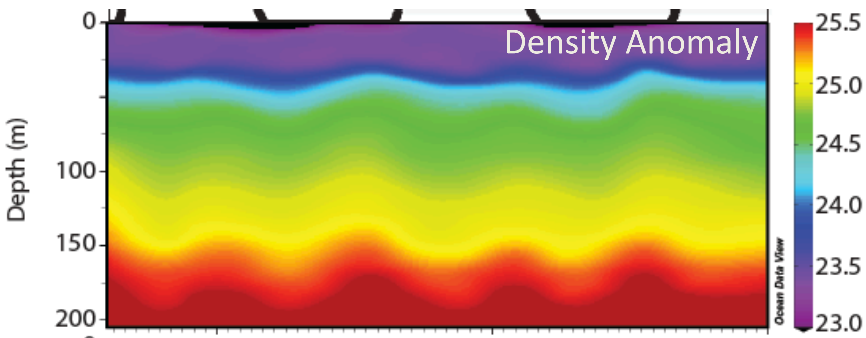
Intermediate KD (shown here)

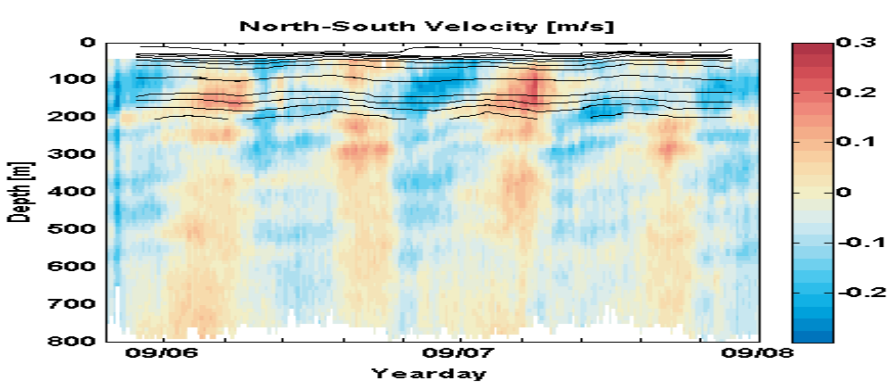
More on water waves?

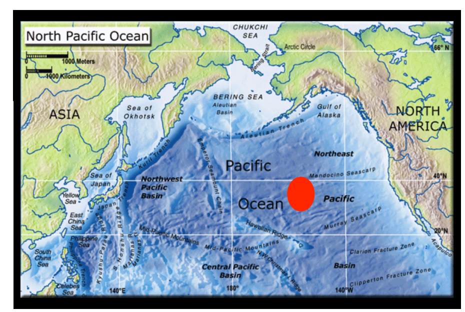
- See e.g. short books by: Pedlosky (2003) *Atmospheric and Oceanic Waves* and Phillips (1966) *Upper Ocean Dynamics*.

Example 2: Internal inertia-gravity waves

- Oscillations of density and momentum in the ocean interior.



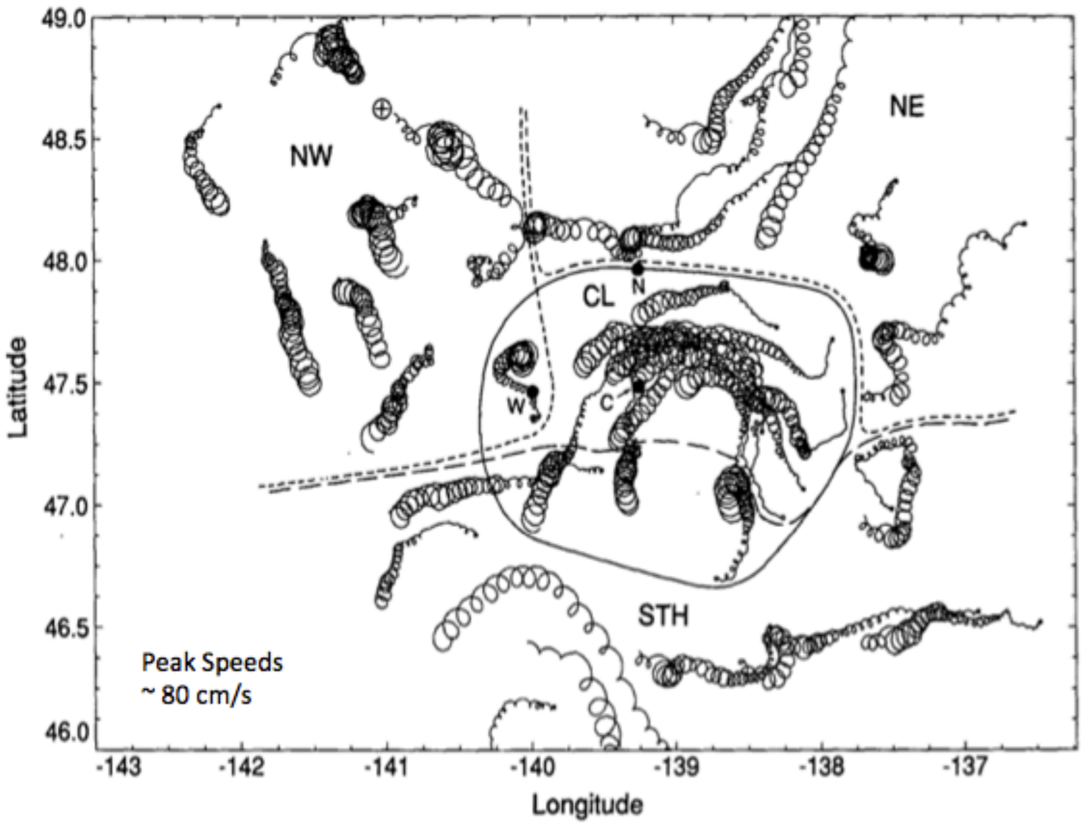




Semi-diurnal internal tide

Example 2:
Internal inertia-gravity waves

- Oscillations of density and momentum in the ocean interior.

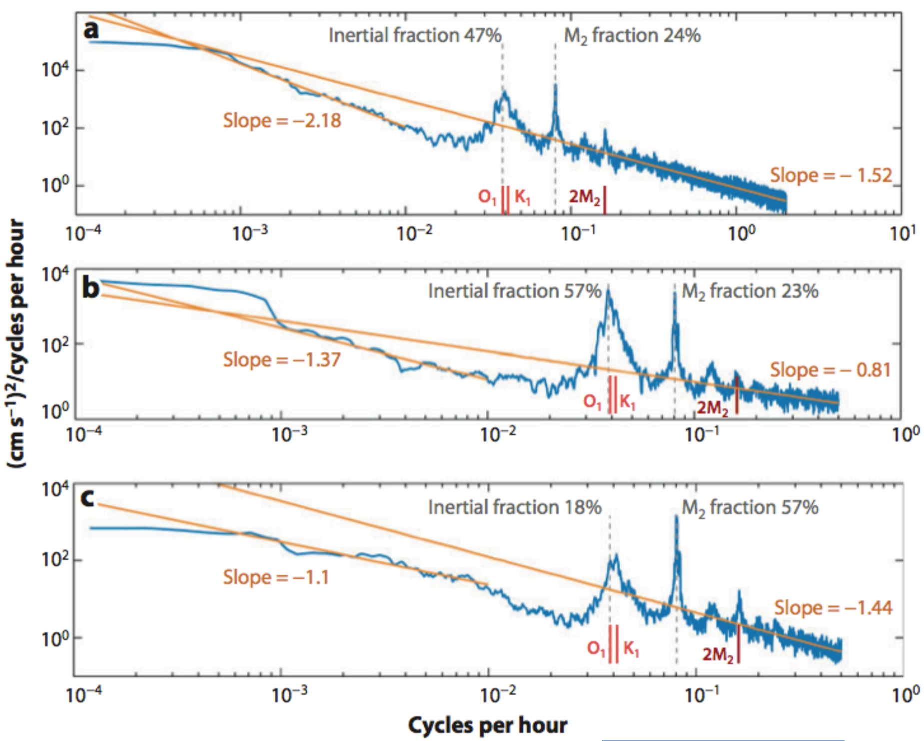


Inertial oscillations at the ocean surface after a storm.

Example 2:
Internal inertia-gravity waves

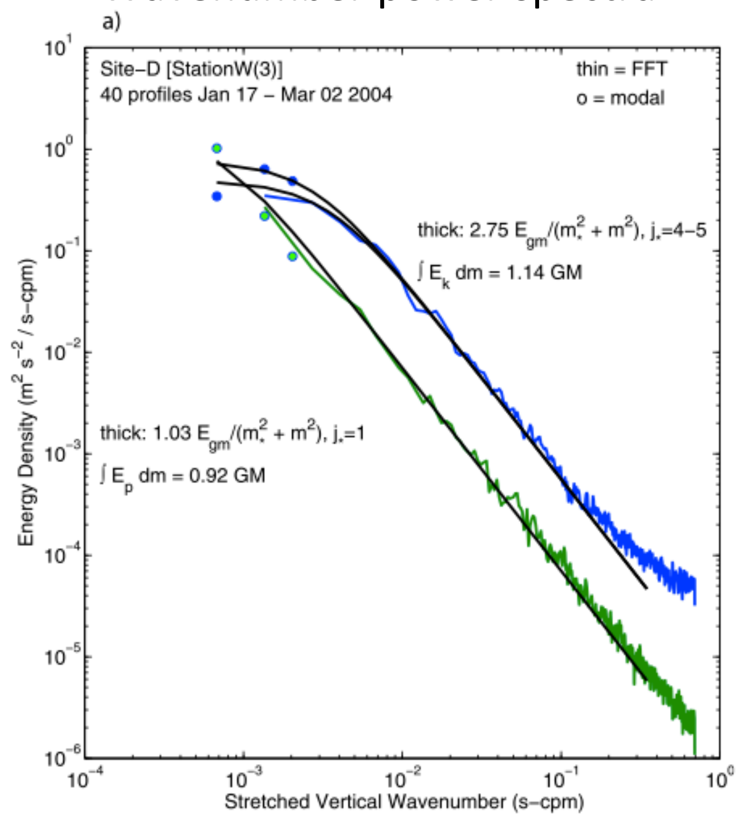
- Oscillations of density and momentum in the ocean interior.

Frequency Power Spectra



Frequency range

Wavenumber power spectra

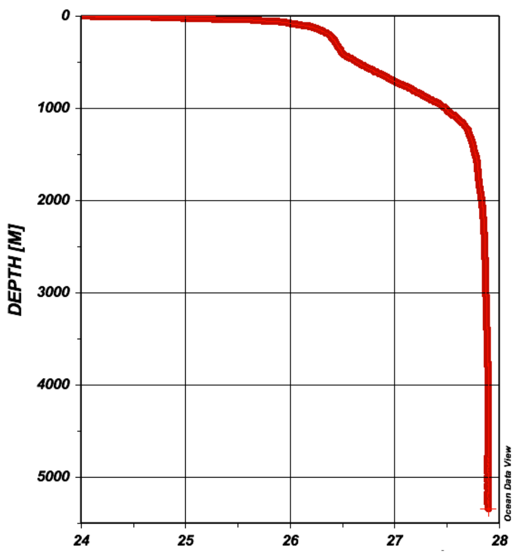


Deriving a Governing Equation

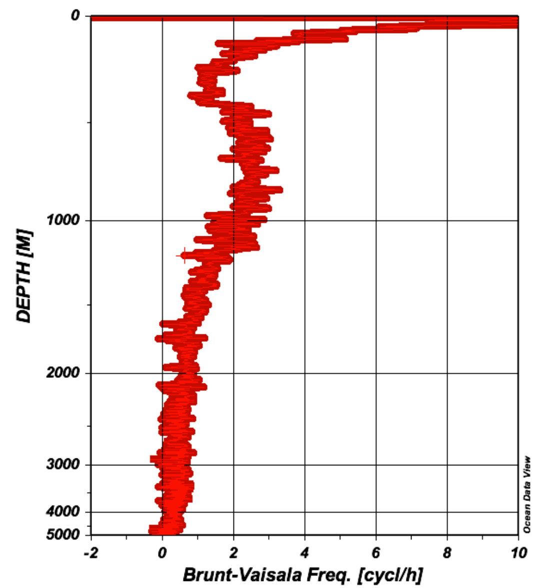
- Step 1: Subtract background density profile.
  - Ocean is stratified and dominant force balance is hydrostatic.

$$0 = -\frac{1}{\rho_0}\frac{\partial \overline{p}(z)}{\partial z} - g\frac{\overline{\rho}(z)}{\rho_0} = -\frac{1}{\rho_0}\frac{\partial \overline{p}(z)}{\partial z} + \overline{b}(z)$$

Density perturbation (from 1000 kg/m^3)



$N = (db/dz)^{1/2}$



Deriving a Governing Equation

- Step 2: make the following scaling assumptions:

Advection is negligible

$$\frac{\tilde{U}\tilde{k}}{\tilde{\omega}} << 1$$

Frictional force is negligible

$$\frac{\nu\tilde{k}^2}{\tilde{\omega}} << 1$$

I have relaxed the assumption that $\omega >> N >> f$

Deriving a Governing Equation

- Step 3: Write down the non-dimensional equations and eliminate small terms. Then re-dimensionalize to obtain:

$$
\begin{aligned}
\frac{\partial u}{\partial t} - f_0 v &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + f_0 u &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \\
\frac{\partial w}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b, \\
\frac{\partial b}{\partial t} + w N^2(z) &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} &= 0,
\end{aligned}
$$

Deriving a Governing Equation

- Step 4: Reduce to one equation in one unknown.

$$\left(\frac{\partial^2}{\partial t^2} + f_0^2\right)\frac{\partial^2 w}{\partial z^2} + \left(\frac{\partial^2}{\partial t^2} + N^2(z)\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = 0$$

Note that we have assumed lateral length scales are sufficiently small that beta (df/dy) is not important.

Here, we will assume that w=0 on top and bottom, i.e. Flat, slippery, rigid lids. One can readily impose other BCs (e.g. rough topography)

Solution

Consider 2-D equations w.o.l.o.g.

$$\left(\frac{\partial^2}{\partial t^2} + f_0^2\right)\frac{\partial^2\psi}{\partial z^2} + \left(\frac{\partial^2}{\partial t^2} + N^2(z)\right)\frac{\partial^2\psi}{\partial y^2} = 0.$$

Insert ansatz: $\psi = \psi_0(z)e^{i(ly-\omega t)}$ $\quad \partial\psi/\partial z = v$ and $\partial\psi/\partial y = -w$

$$\frac{d^2\psi_0}{dz^2} + m^2\psi_0 = 0,$$

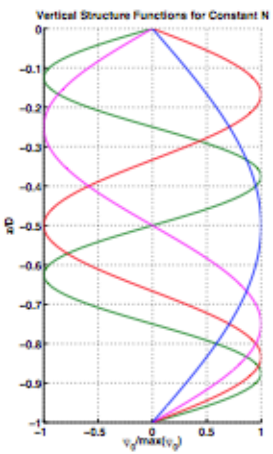
Solve ODE for vertical structure:

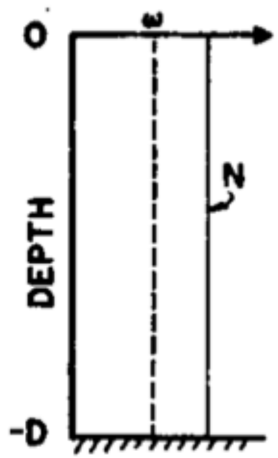
$$m^2 = l^2\frac{(N^2(z) - \omega^2)}{(\omega^2 - f_0^2)},$$

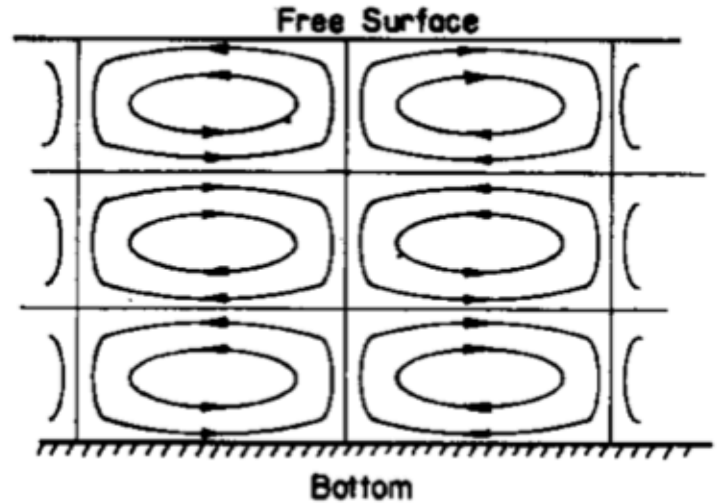
$\psi_0(z) = \sin(mz)$, where $m = \pi j/D$ with $j = 1, 2, ...$

for constant N. Arbitrary structure functions can be obtained numerically.

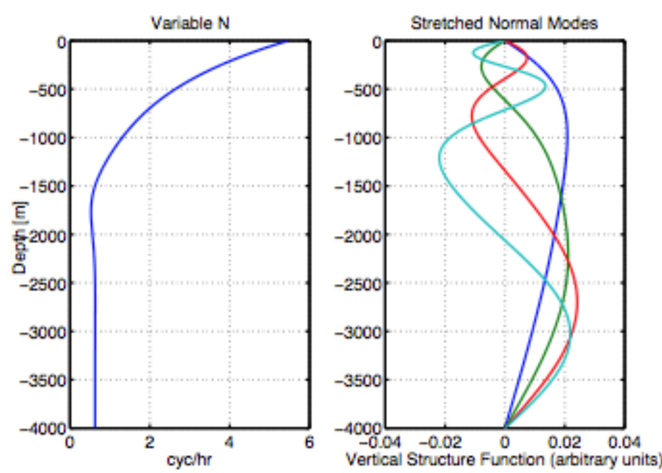
Example Vertical Structure Functions







At constant N:



High wavenumbers are unaffected by boundaries and locally at constant N

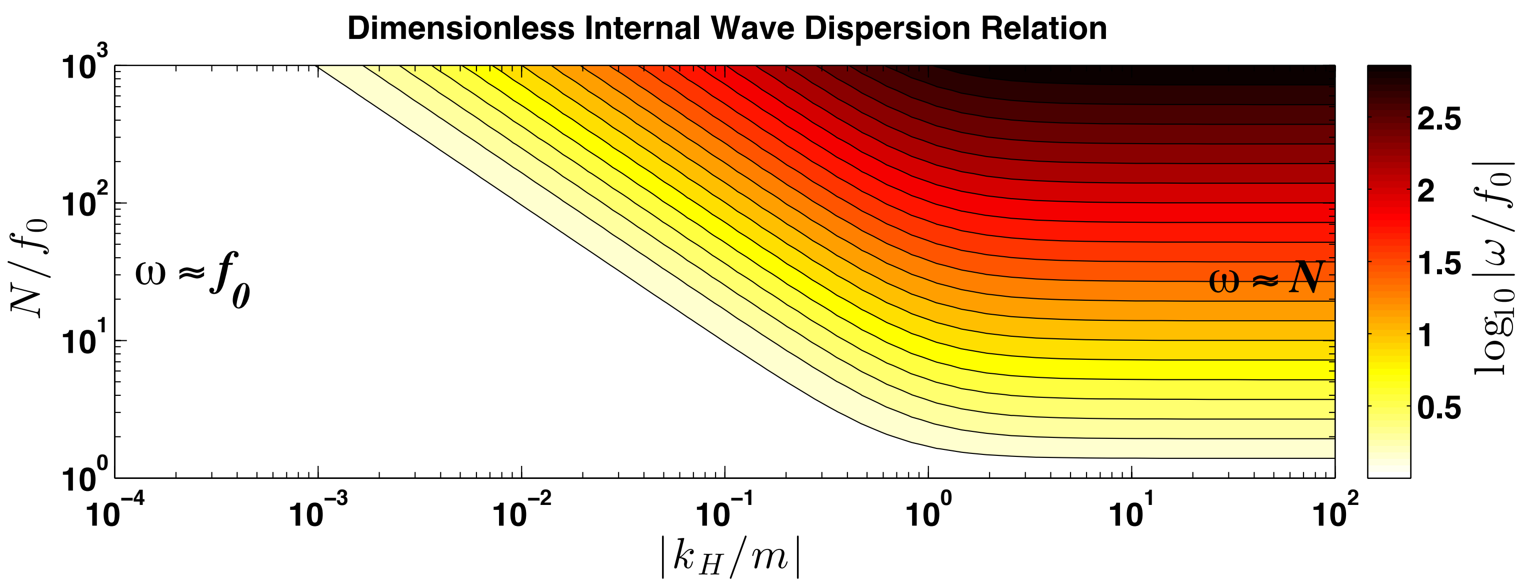
=> We use the pure plane wave ansatz:

$$\psi = Re(\psi_0 e^{i(kx+ly+mz-\omega t)})$$

$$\omega/f = \pm\sqrt{\frac{1+\frac{N^2}{f_0^2}\frac{k_H^2}{m^2}}{1+\frac{k_H^2}{m^2}}}$$

$$f_0 < \omega < N$$

$$k_H = \sqrt{k^2+l^2}$$

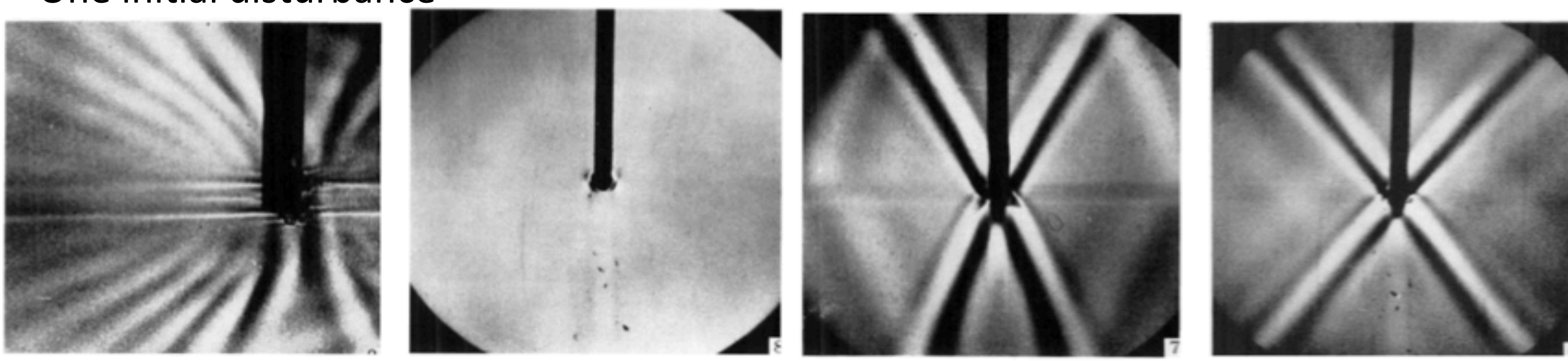


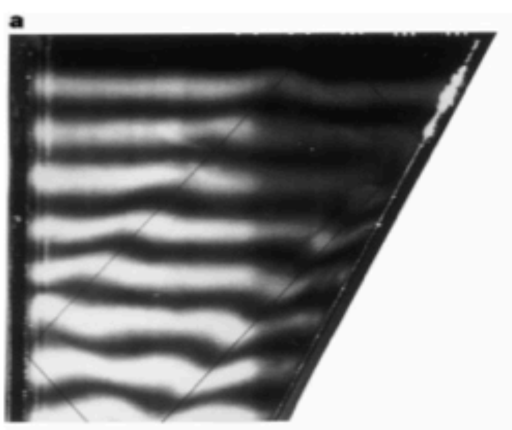
Interesting property

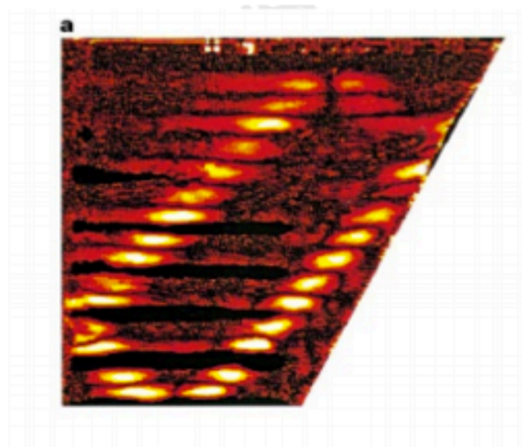
- Aspect ratio fixed by frequency & background (N,f). Mowbray and Rarity (1967), Maas et al. (1997), Gautiaux et al. (2006)

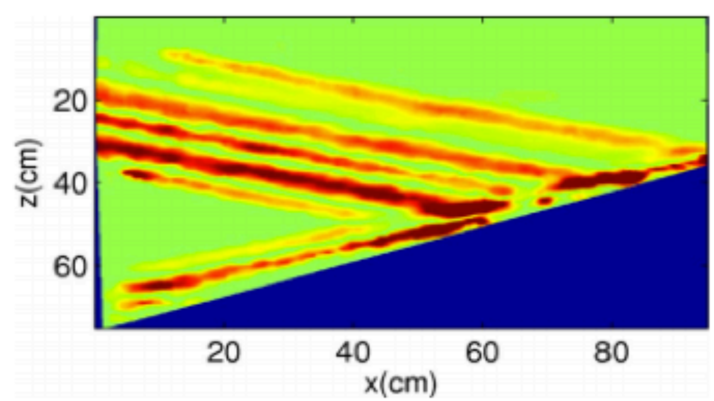
$$\omega = 1.1N, 0.9N, 0.7N$$

One initial disturbance









More?

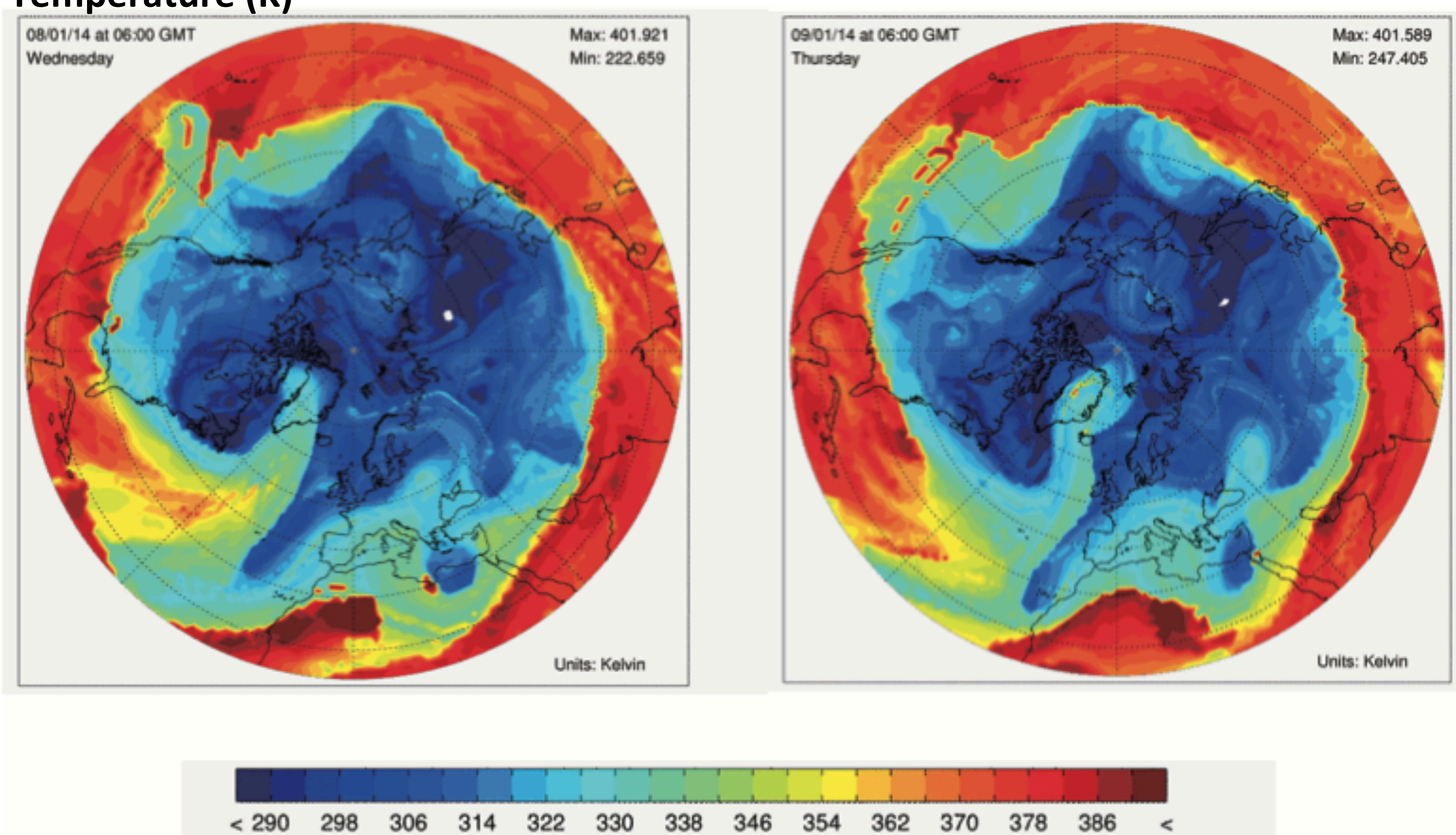
- You’re in luck! I’m going to discuss internal waves in an inhomogeneous medium tomorrow (although the ideas will apply to all types of waves).
- See Pedlosky (2003), Phillips (1966), Lighthill (1978) *Waves in fluids*, and Munk (1981), *Internal waves and small scale processes*

Example 3:
Rossby Waves (Potential Vorticity Waves)

$$\omega << f_0$$

Potential Temperature (K)

(Hoskins 2015)

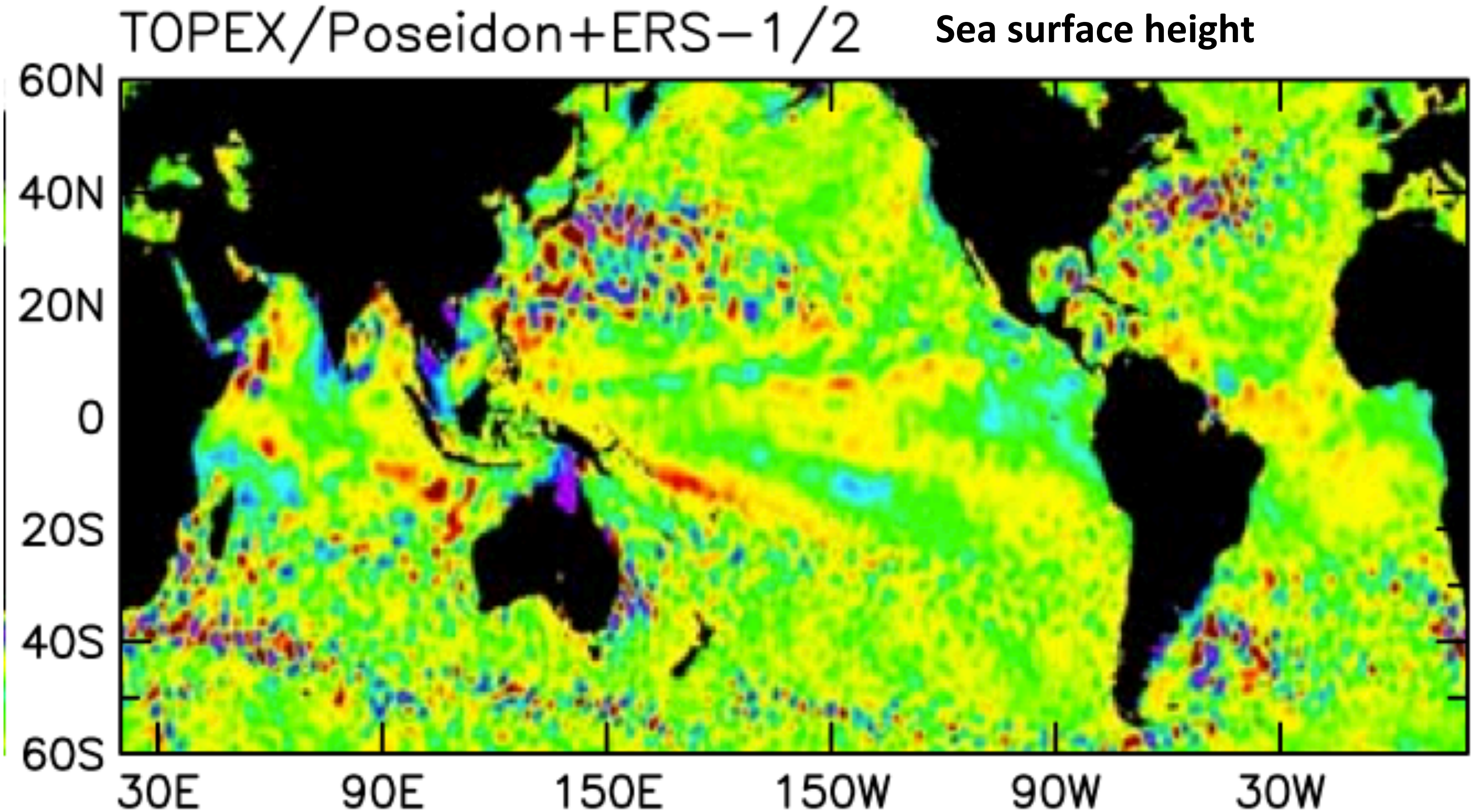


Example 3:

Rossby Waves (Potential Vorticity Waves)

$$\omega << f_0$$

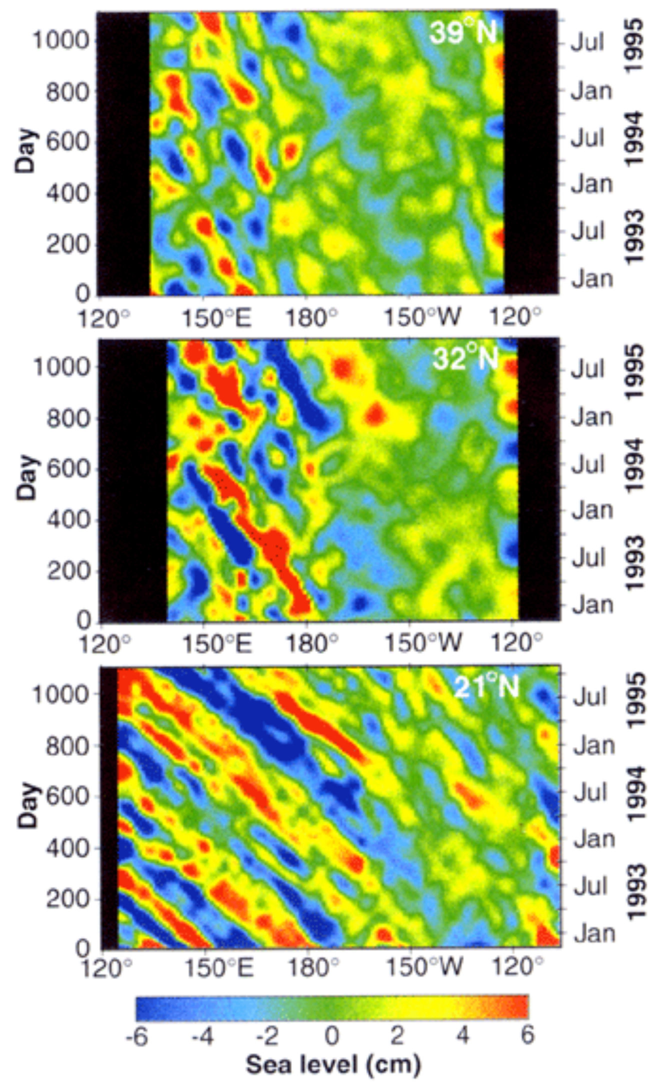
(Chelton et al. 2007)



Important question

Chelton and Schlax (1996)

- How linear are these motions?



Deriving a governing equation

- Step 1: Subtract hydrostatic balance.
- Step 2: Make scaling assumptions:

$\tilde{\omega} \sim \tilde{U}/\tilde{L} << \tilde{f}$ $\quad \tilde{H} << \tilde{L} \quad$ $\tilde{W} \sim \tilde{H}\tilde{U}/\tilde{L}$

$\mathbf{f} = (f_0 + \beta y)\mathbf{k}$ $\quad$ $\tilde{\beta} \sim \tilde{U}/\tilde{L}^2$

$Ro = \tilde{U}/\tilde{f}\tilde{L} << 1$, where Ro is known as the *Rossby number*

Deriving a governing equation

- Step 3: write non-dimensional equations where we replacing e.g. $\tilde{U}\hat{u} = u$

$$Ro\left(\frac{\partial \hat{\mathbf{u}}_{\mathbf{H}}}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}}_{\mathbf{H}}\right) + \hat{\mathbf{f}} \times \hat{\mathbf{u}}_{\mathbf{H}} = -\nabla_H \hat{p},$$

$$\frac{RoH}{L}\left(\frac{\partial \hat{w}}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \nabla \hat{w}\right) + \frac{\partial \hat{p}}{\partial \hat{z}} = \hat{b},$$

$$Ro\left(\frac{\partial \hat{b}}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \nabla \hat{b}\right) + \hat{w} Bu = 0,$$

$$\nabla \cdot \hat{\mathbf{u}} = 0,$$

$$\hat{f} = \hat{f}_0 + Ro\hat{\beta}\hat{y}$$

where $Bu = \tilde{N}^2 \tilde{H}^2 / \tilde{f}_0^2 \tilde{L}^2$ is known as the Burger number

Deriving a governing equation

- Step 4: Consider an asymptotic expansion in Rossby number

$$\begin{aligned}\hat{\mathbf{u}} &= \hat{\mathbf{u_0}} + Ro\hat{\mathbf{u}}_\mathbf{1} + ... \\ \hat{p} &= \hat{p_0} + Ro\hat{p}_1 + ... \\ \hat{b} &= \hat{b_0} + Ro\hat{b}_1 + ...\end{aligned}$$

Lowest order: geostrophic and hydrostatic balance (no time evolution and horizontally non-divergent)

$$\hat{\mathbf{f_0}} \times \hat{\mathbf{u_0}} = -\nabla \hat{p}_0 \qquad \hat{w}_0 Bu = 0.$$

First order: time evolution equations for Rossby waves

$$\frac{\partial \hat{\mathbf{u}}_{\mathbf{0H}}}{\partial \hat{t}} + \hat{\mathbf{u}}_{\mathbf{0H}} \cdot \nabla \hat{\mathbf{u}}_{\mathbf{0H}} + \hat{\beta}\hat{y}\hat{\mathbf{k}} \times \hat{\mathbf{u}}_{\mathbf{0}H} + \hat{\mathbf{f}} \times \hat{\mathbf{u}}_{\mathbf{1H}} = -\nabla_H \hat{p}_1$$

$$\frac{\partial \hat{b_0}}{\partial \hat{t}} + \hat{\mathbf{u_0}} \cdot \nabla \hat{b}_0 + \hat{w}_1 Bu = 0. \qquad \frac{\partial \hat{u}_1}{\partial \hat{x}} + \frac{\partial \hat{v}_1}{\partial \hat{y}} + \frac{\partial \hat{w}_1}{\partial \hat{z}} = 0$$

Deriving a governing equation

- Step 5: Reduce the system to one dimensionless equation

Consider vorticity eqn:
$\zeta = \partial v_0/\partial x - \partial u_0/\partial y$

$$\frac{\partial \hat{\zeta}_0}{\partial \hat{t}} + \mathbf{\hat{u}_0} \cdot \nabla \hat{\zeta}_0 + \hat{v}_0 \hat{\beta} = \hat{f}_0 \left(\frac{\partial \hat{w}_1}{\partial \hat{z}} \right)$$

Then use buoyancy evolution equation to replace w_1 with b_0, use zeroth order hydrostatic relation to replace b_0 with p_0, and finally:

replace $\hat{p}$ with $\hat{\psi}$ $(\hat{f}_0 \hat{\psi} = \hat{p}_0)$

Stream function:

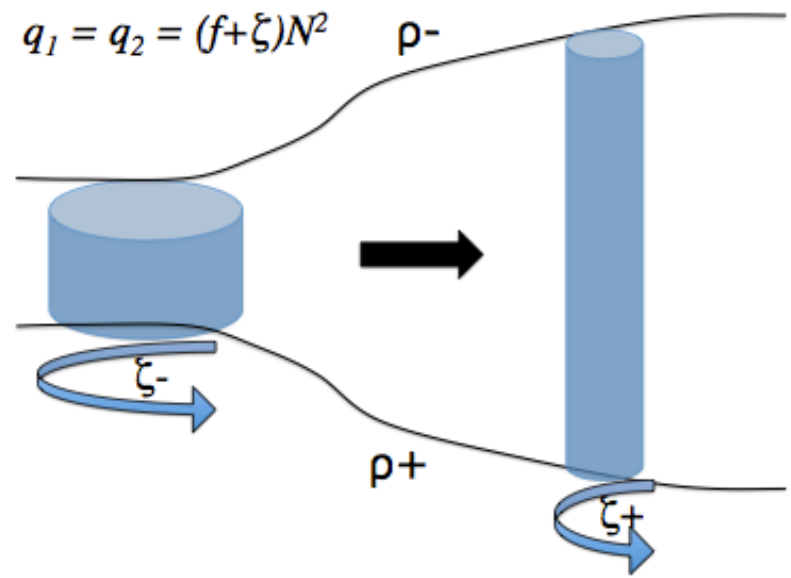
$$\frac{\partial \hat{\psi}}{\partial \hat{y}} = -\hat{u},$$
$$\frac{\partial \hat{\psi}}{\partial \hat{x}} = \hat{v}.$$

$$\frac{D_0}{D\hat{t}} \left[\Delta \hat{\psi}_0 + \hat{\beta} \hat{y} + \hat{f}_0{}^2 \left(\frac{\partial}{\partial \hat{z}} \left(\frac{1}{Bu} \frac{\partial \hat{\psi}_0}{\partial \hat{z}} \right) \right) \right] = 0.$$

where $D_0/D\hat{t} = \partial/\partial \hat{t} + \mathbf{\hat{u}_0} \cdot \nabla$

Quasi-geostrophic Potential Vorticity (PV) Conservation

A stratification (or layer-thickness) weighted vorticity



The dimensional equation is:

$$\frac{Dq}{Dt} = 0,$$

$$q = \Delta\psi + f_0 + \beta y + f_0^2 \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial \psi}{\partial z}\right)$$

where q is known as the *quasi-geostrophic potential vorticity*

Ducted Rossby (PV) wave solutions in continuous stratification

Linearized QGPV conservation

$$\frac{\partial}{\partial t}\left(\Delta\psi + f_0^2 \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial \psi}{\partial z}\right)\right) + \beta\frac{\partial \psi}{\partial x} = 0$$

Insert ducted plane-wave ansatz:

$$\psi = \psi_0(z) e^{i(kx+ly-\omega t)}$$

Solve ODE for vertical structure:

$$\left(\beta k + \omega K^2\right)\psi_0(z) = \omega \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2(z)}\frac{\partial \psi_0(z)}{\partial z}\right)$$

If we assume N is constant, and that $\partial\psi/\partial z = 0$ at the top and bottom

Solution: $\cos(mz)$, where $m = j\pi/D$ with $j = 1, 2, \ldots$

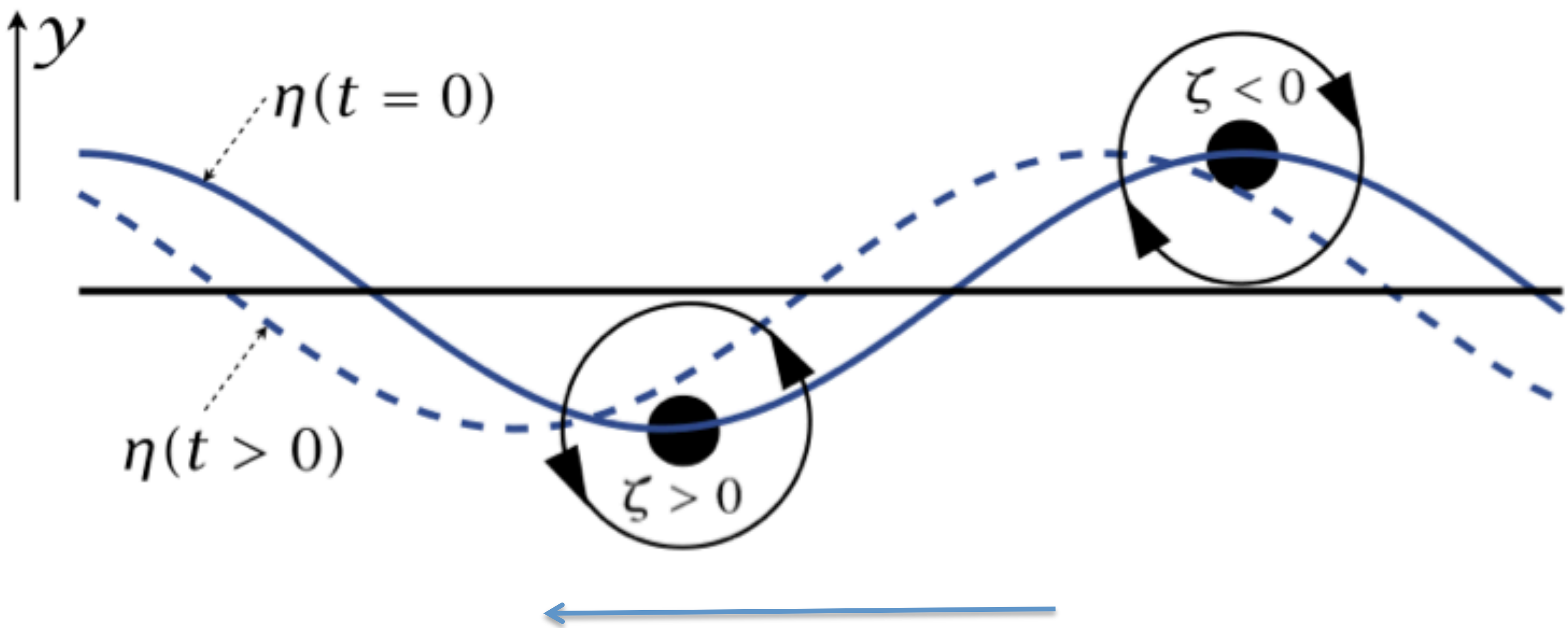
Dispersion relation:

$$\omega = \frac{-\beta k}{K^2 + m^2 \frac{f^2}{N^2}}$$

Dynamical Sketch

$f = f_0 + \beta y$

PV gradient



Westward phase propagation

Vallis (2006)

An interesting result

- The zonal phase speed ω/k always points from east to west.

$$\omega = \frac{-\beta k}{K^2 + m^2 \frac{f^2}{N^2}}$$

- Back of the envelope estimate of phase speed in the mid-latitude ocean:

$$\text{For } 20^o \text{ N}, \beta \approx 2 \times 10^{-11} \text{ s}^{-1} \text{ m}^{-1}$$
$$m \approx 2\pi/5000 \text{ m}^{-1}, \text{ and } f_0^2/N^2 \approx 10^{-4}$$

phase speed, ***ω/k ~ 10 km/day***

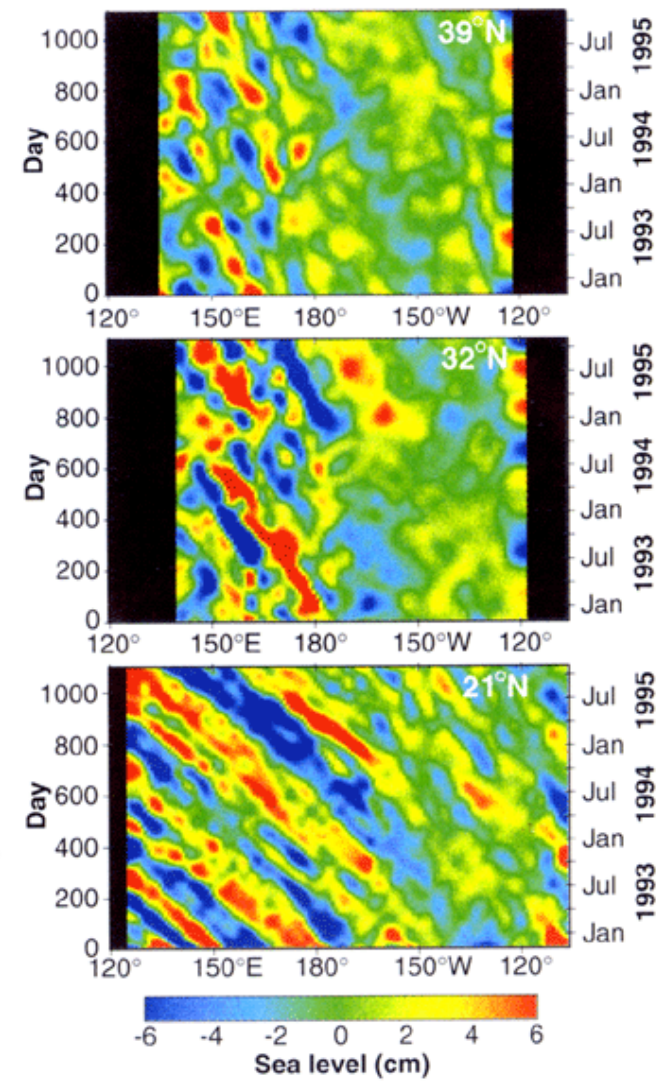
Important question

Chelton and Schlax (1996)

- How linear are these motions?

Note westward phase propagation.

@20°N, phase lines propagate about 3000 km in 300 days ***(~10 km/day)***



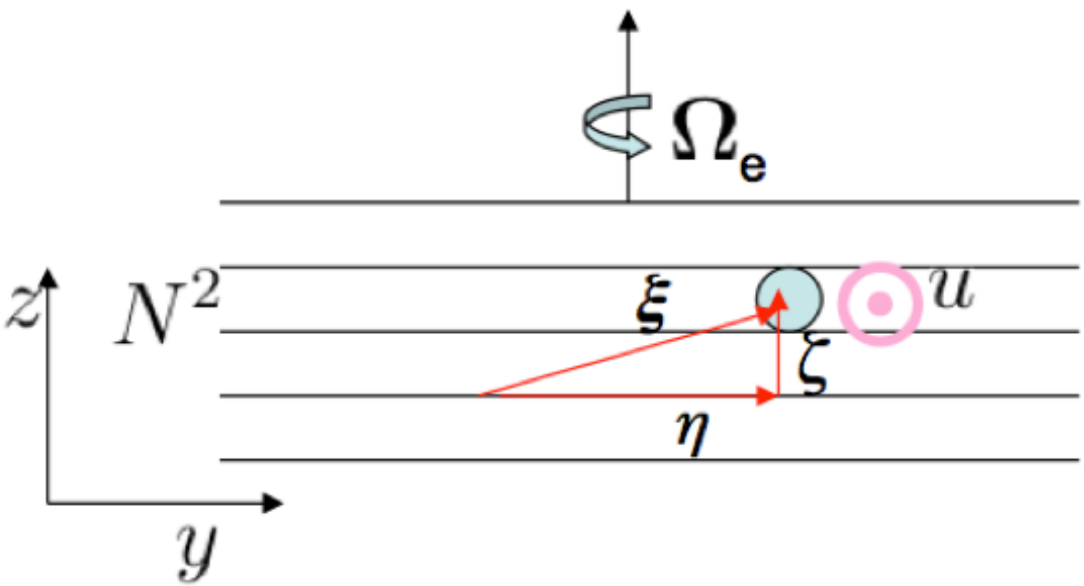
More?

- Vallis (2006), Atmospheric and Oceanic Fluid Dynamics, Pedlosky (2003), and Pedlosky (1986), GFD.
- Also Hoskins et al. (1985), *On the use and significance of isentropic potential vorticity maps.*

Some other topics

- Lagrangian/Conservation Law perspective for internal waves
- Wave propagation in an inhomogeneous medium
  - Trapping, amplification, & wave breaking

Lagrangian parcel analysis



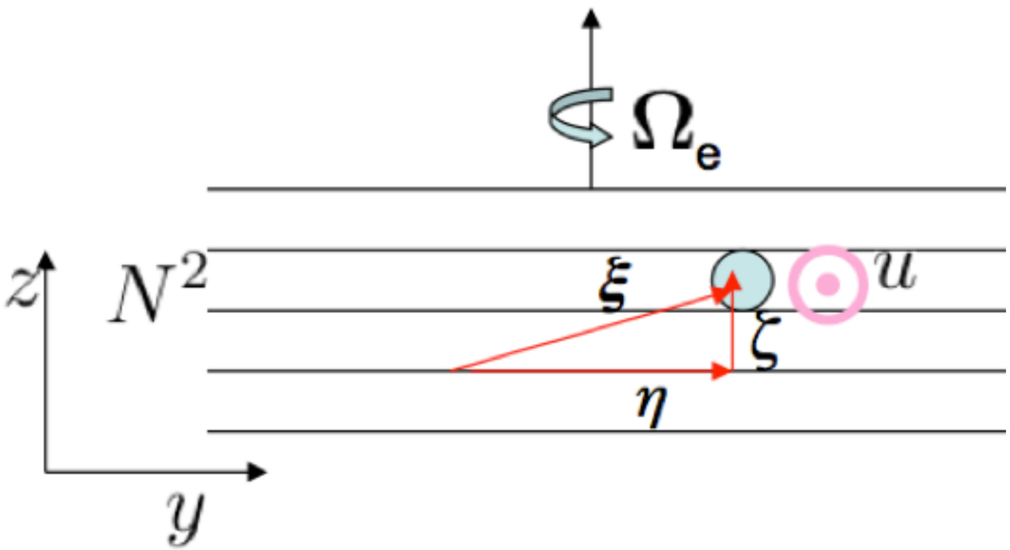
Assuming that the motion adapts immediately to the background pressure

$$
\begin{aligned}
\frac{\partial u}{\partial t} - f_0 v &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x},\\
\frac{\partial v}{\partial t} + f_0 u &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y},\\
\frac{\partial w}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b,\\
\frac{\partial b}{\partial t} + wN^2(z) &= 0,\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial u}{\partial z} &= 0,
\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
\frac{DM_x}{Dt} &= 0, & \frac{Dw}{Dt} &= b',\\
\frac{DM_y}{Dt} &= 0, & \frac{Db}{Dt} &= 0,
\end{aligned}
$$

$M_x = u(x, y, z, t) - f\eta$ is conserved, $M_y = v(x, y, z, t) + f\xi$

$D\zeta/Dt = w$, $D\eta/Dt = v$ and $D\xi/Dt = u$

Lagrangian parcel analysis



forces as a function of displacement,

$$F_x = \frac{Du}{Dt} = +fv = -f^2\xi,$$
$$F_y = \frac{Dv}{Dt} = -fu = -f^2\eta,$$
$$F_z = \frac{Dw}{Dt} = b' = -N^2\zeta,$$

Imply the following governing equation

$$\frac{D^2|\boldsymbol{\xi}|}{Dt^2} = -f^2|\boldsymbol{\xi}_H|\cos(\theta) - N^2|\zeta|\sin(\theta) = -|\boldsymbol{\xi}|\left(f^2\cos^2(\theta) + N^2\sin^2(\theta)\right)$$

$$\boxed{\omega^2 = f^2\cos^2(\theta) + N^2\sin^2(\theta),}$$

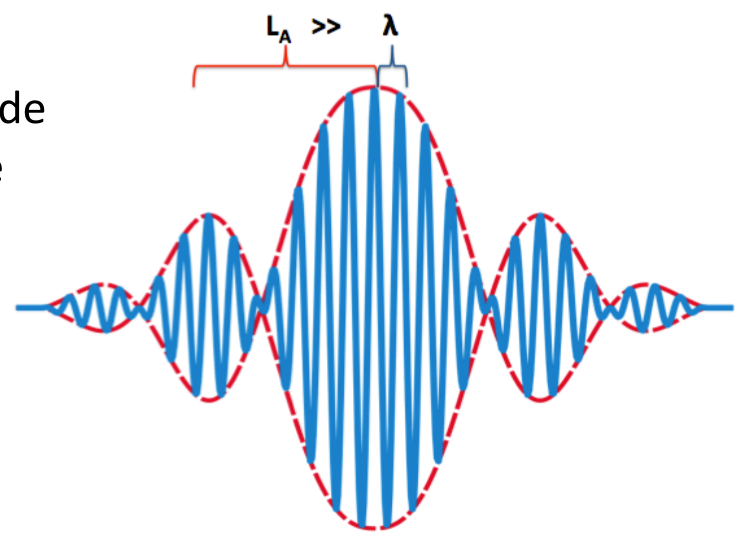
for the magnitude of the displacement $|\boldsymbol{\xi}|$ at angle $\theta = \tan^{-1}(\sqrt{|\xi|^2 + |\eta|^2}/|\zeta|)$

Teaser: Waves in an inhomogeneous medium

- Inhomogeneity in the medium/wavefield is a necessary (but not sufficient) condition for understanding waves in the real ocean

Idea is to generalize plane wave solution so that it is modulated by an envelope/amplitude function that varies slowly in time and space compared to frequency and wavelength respectively.

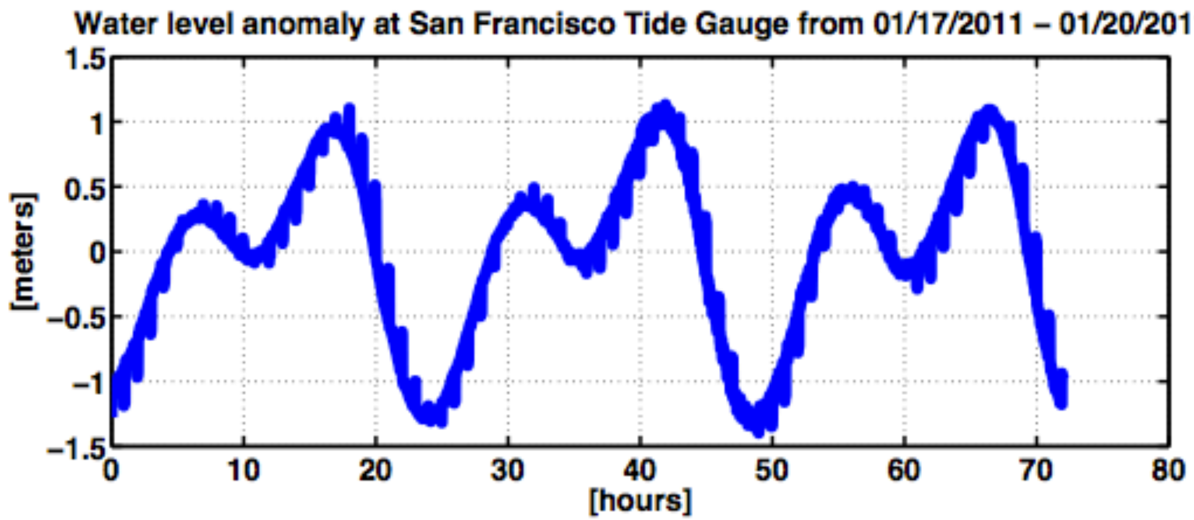
$$\psi = Re(\psi_0(x, y, z, t)e^{i(\alpha(x,y,z,t)})$$



Conclusions

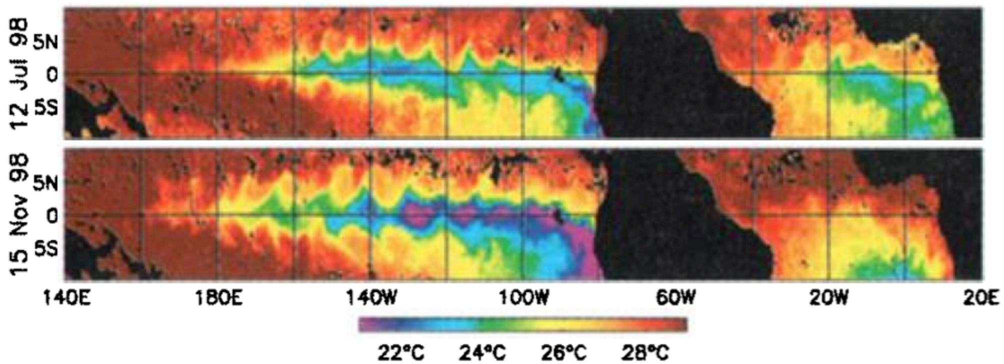
- Fundamental mathematical tools for wave theory are broadly applicable.
- However, a detailed analysis can still yield surprising results in specific cases.
- This is just the beginning!
  - Wave theory is a foundational concept in GFD and plays an important role in our understanding of both the atmospheric and oceanic general circulation.
- Just to emphasize this:
  - Two important examples that I haven't discussed

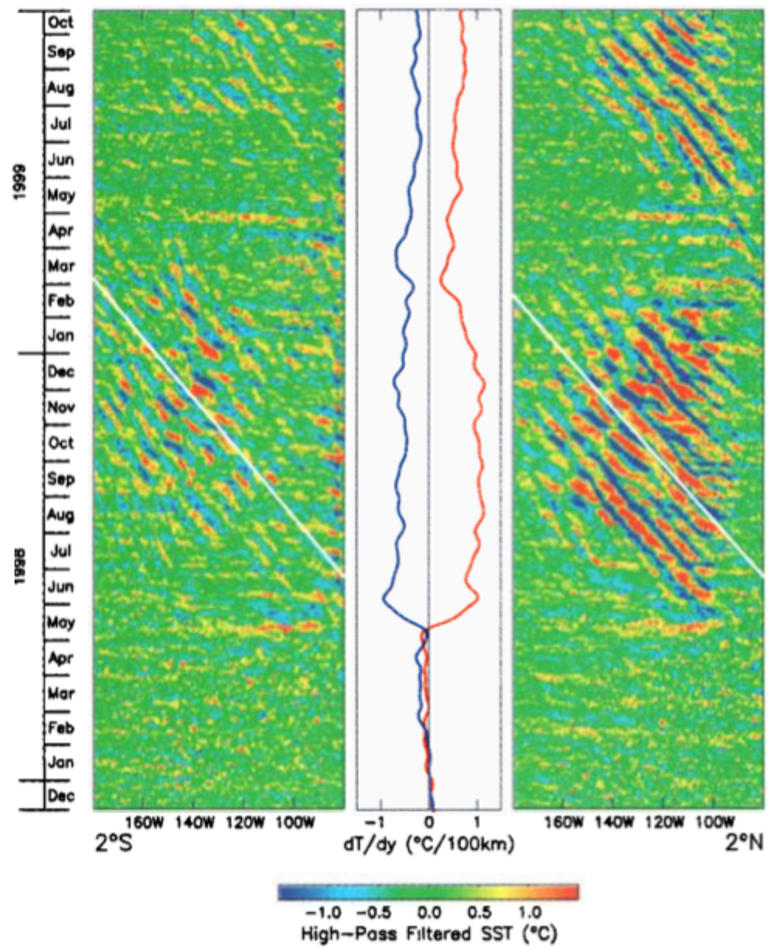
Tides


Water level anomaly at San Francisco Tide Gauge from 01/17/2011 – 01/20/2011
[meters]
[hours]
1.5
1
0.5
0
−0.5
−1
−1.5
0
10
20
30
40
50
60
70
80

Tropical Instability Waves

Snapshot of sea surface temperature





Westward propagation along the equator.
Not pure Rossby waves.

Outline #2

- Wave energy propagation in an inhomogeneous medium.
- Critical layers and turning points
  - Wave trapping and energy convergence (shoaling)
- Wave/turbulence transitions, wave breaking
  - Surf zones
  - Mean circulations driven by dissipating waves.
- Balance and imbalance: if it wiggles, is it a wave?
  - wave/vortex decompositions
  - pathways for energy exchange between balanced and unbalanced flows
    - Link to KE budget of the general circulation