Optimal Strategies for Stochastic Models of Running Performance

Daniel Whitt

May 3, 2009

1. Introduction

It is well known that the selection of an optimal pacing strategy is critical to success in long-distance running. Most experts agree that these events, which range between 1500 meters on the track and 26.2 miles on the road, require a fairly even distribution of work throughout the event. However, the physiological triggers which cause a runner to regulate his pace remain elusive today (Davis and Bailey 1997).

As our understanding of human physiology becomes increasingly nuanced, it becomes more difficult to bridge the gap between application and theory. Every new study adds more detail to our knowledge about the exercising human body. As a result, increasingly complicated models are required to accurately model these details. In most cases, however, complicated models, although they may predict performances well, do not illustrate the real behaivior in an intuitive way. And, moreover, despite the improved accuracy of these models, there remains some uncertainty in running performance. For these reasons, and because simpler models remain good approximations of human performance, we will take a step back from a physiological perspective and examine some more primitive models. It will reflect our uncertain understanding of human fatigue, but avoid getting any deeper into the physiology. Although the physiological details are necessary to understand the actual workings of the body, they do not aid with application.

2. The Deterministic Theory of Running

In 1973 Joe Keller introduced a simple theory of competitive running (Keller 1973). The paper provides an optimal pacing strategy for running races of varying distances and determines the values of four physiological constants by comparing the model with the world records. After a reformulation, the optimization problem was solved analytically with variational methods. The results were clear and easily applicable. Keller determined theoretical optimal pacing strategies for all races over standard track distances (50m-10000m) and the values of the four constants. I will now give a brief outline of Keller's model and results. See his 1973 and 1974 papers on the subject for more details.

Following Keller's construction, we relate the length of the race, D, the time taken to run the race, T, and the instantaneous velocity at any time during the race, $v(t)$ by the following two equations:

$$D = \int_0^T v(t)dt \quad \text{and} \quad T \geq 0$$

Then we use Newton's Law to relate the instantaneous propulsive force per unit mass, $f(t)$, to the acceleration and a single resistive force per unit mass, $v(t)/\tau$.

$$\frac{dv}{dt} + \frac{v}{\tau} = f(t)$$

$f(t)$, the propulsive force, is under the control of the runner. The runner's goal is to select the function in such a way as to minimize T subject to the constraint that at no time t may $f(t)$ exceed a maximum propulsive force per unit mass which we will denote F:

$$f(t) \leq F$$

There is, however, an additional constraint upon $f(t)$ because the energy supply of the runner, $E(t)$, must always be greater than 0.

In this simplified model there are only two sources of energy. The first is a small, finite, but immediately available store in the muscles, E_0. The second is a source of energy production in the body which we assume has a constant rate per unit mass, σ. One can choose to view σ in a variety of ways to gain an intuitive understanding of the physical processes. Or, think of it, as Keller did, as proportional to the rate at which oxygen can be supplied to the working muscles. As a result, the rate of change in the available energy is determined by the rate in, σ, and the rate out, the instantaneous power expended, $P(t) = f(t)v(t)$. We can express this energy rate relation:

$$\frac{dE}{dt} = \sigma - f(t)v(t)$$

Keller's solution has two cases. For $T \leq T_c$, a critical distance, the optimal force function is $f(t) = F$ and the velocity is given by

$$v(t) = F\tau(1 - e^{-t/\tau})$$

In this case, the race is so short that the runner is unable to expend the available energy before reaching the finish line even with maximal force applied for the entirety of the race. On the otherhand, if $T > T_c$ the runner would be out of energy before the finish if he used maximal force throughout the race. Hence, the runner would need to optimize his energy use so as to minimize the time and remain above zero energy. In this case, $v(t)$ initially increases with maximum force applied until an optimal cruising speed is reached at which point the velocity remains constant until just before the finish when it drops off right before the finish line with the energy reserves remaining constant at zero and velocity decreasing to the equilibrium velocity where the rate of change of energy is zero. After fitting the model with world record performances, by minimizing the squared error between the model predictions and the actual records, the values for the constants are obtained.

The table of physiological constants appears below.

Constant	Value
σ	41.5 Joules/(kg*sec)
E_0	2406 Joules/kg
τ	0.892 seconds
F	$12.2m/s^2$
D_c	$291m$

Table 1: Physiological Constants obtained from Keller's model. A version of this chart appears in Keller's 1973 paper.

For evidence in support of Keller's model, consider some of my recent race performances in table 2. Using a couple of standard metrics I ranked the races which I ran last year over various long distances on the track. As the table indicates, the best performances came off relatively slower early paces as can be seen from the ratio of the first half to the second half. In this column, numbers greater than 1 indicate a slower second half than a first half and numbers less than 1 indicate a faster second half. My results are typical for an aspiring runner. I frequently go out hoping to run faster than I am able. Then I end up with a relatively inferior performance. Of course, these results only represent a trial of one. Moreover, I faced opposition in these races so no general conclusions can be drawn. An entire season of this data is not usually available for one runner, though, so it is still useful to have a glimpse of how one runner's different performances compare over a short time period.

3. Changing Views on Human Fatigue

Since Keller developed his model, our understanding of human physiology has changed significantly. Despite many advances, however, no one has shown conclusively why runners become fatigued and ultimately regulate their pace. In Keller's model it is the limiting constant rate of

Date	Length (m)	Time (min.)	First 1/2	Second 1/2	Ratio	Score	Rank
Feb 08	3000	8.617	4.25	4.367	1.028	846	3.5
Feb 08	5000	14.98	7.335	7.645	1.042	811	3.5
Feb 08	3000	8.7	4.2	4.5	1.07	822	5
Apr 08	5000	14.91	7.5	7.41	.988	824	1.5
Apr 08	5000	15.017	7.29	7.781	1.067	806	5
May 08	10000	31.15	15.45	15.7	1.016	860	1.5

Table 2: **My 2008 track performances.** The ratio is the time for the second half divided by the time for the first half. The Score comes from the IAAF scoring tables. The rank is determined by ranking the races 1 (the best) to 6 (the worst) using Greg McMillan's Running Calculator (http://www.mcmillanrunning.com) and then doing the same for the IAAF scores and taking the arithmetic average of the two rankings.

oxygen delivery to the working muscles (σ) which forces a regulation of pace in the longer races. The theory that oxygen delivery limits endurance performance has been popular in physiology for many years. However, it has been shown to be either overly simplified or completely wrong in recent years (Noakes 2003, Chapter 2). One popular alternative theory is proposed by Tim Noakes, an exercise physiologist at the University of Capetown in South Africa. This theory suggests that the subconcious brain plays a much more critical role in regulating pace than actual limitations in oxygen delivery capability (Noakes 2003). This suggests that, although we can measure all the physiological data at the peripheral locations in the body, we will have to change the approach of our research to really understand fatigue. In light of this theory, Keller's model seems too simple. However, because it makes a good first approximation with a very simple construction, and because no one has yet developed a really good model for human fatigue, it is still quite relevant.

The difficulties with Keller's model are well known and have been addressed by many others in attempts to better model world record performances and predict the limits of human performance. For example, many have observed that since Keller's model has a 'minimum speed limit', at which the runner can continue indefinitely powered by breathing alone. Hence, the longer so called ultra distances are not modeled well because the fatigue factors in ultra racing are neglected. Modeling advances have helped to fix this problem and can now predict the newly understood fatigue factors at distances longer than 10000m. For examples recent discussions on the subject, see W. Woodside's 1991 and W.G. Pritchard's 1993 papers. My analysis, however, is concerned with the optimal strategies, not predicting records. As a result, my approach is a little different from what has been done before.

Little scrutiny has been given to this second use of Keller's theory, determining the optimal strategy for a race. Keller noted in his 1973 paper that the optimal pacing strategy predicted by the theory is rarely employed in the long distance races. Competitors are often wary of their opponents and reluctant to take the lead in competitions. The result is usually a furious finishing sprint after a moderate pace during the bulk of the event. However, in paced world record attempts, where competition is limited primarily to the clock, the best races are run with an even or slight negative split, meaning the second half is equal to or slightly faster than the first half. This observation is roughly in agreement with Keller, but velocity profiles of these races usually differ dramatically from Keller's model. Consider, for example, three velocity profiles depicted in figures 1 and 2. The first, in figure 1, is Keller's optimum pacing strategy for a race of about 45 seconds in length ($\sim$ 400m). The general form is typical of races longer than T_c in Keller's model.

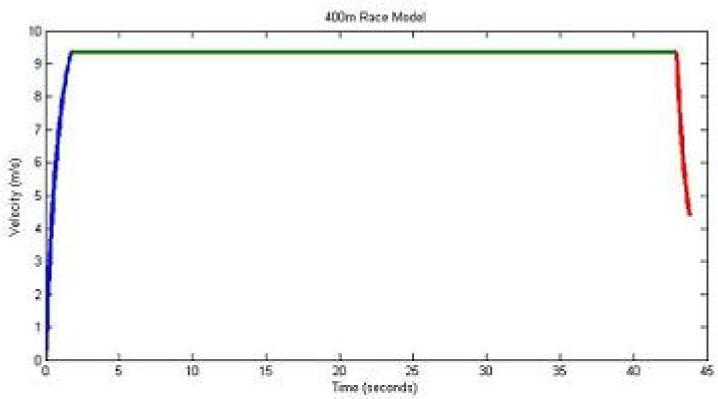

Figure 1: A plot of the velocity as a function of time as predicted for the 400m, a race longer than the critical distance.

The two plots appearing in figure 2, are actual performances. One is a club level cyclist performing a 20km time trial where the goal is to cover the 20km distance in a minimum time and the second is a 1998 world record performance over 10000m on the track by Haile Gebreselassie of Ethiopia where the goal is also to cover the distance in the least time.

Although only a few real examples have been given, they are typical. They demonstrate what regular observers of endurance performance take as given. In the macroscopic picture, the race is evenly paced. In the world record 10k, the first half and second half were less than one second apart. However, during the final ten percent of a race, a marked acceleration frequently occurs, regardless of the level of the athlete. This acceleration is in stark contrast to the prediction of Keller's model, which suggests, if anything, a drop-off at the very end. In Keller's theory, these athletes are not racing to their ultimate potential. The question then becomes, why?

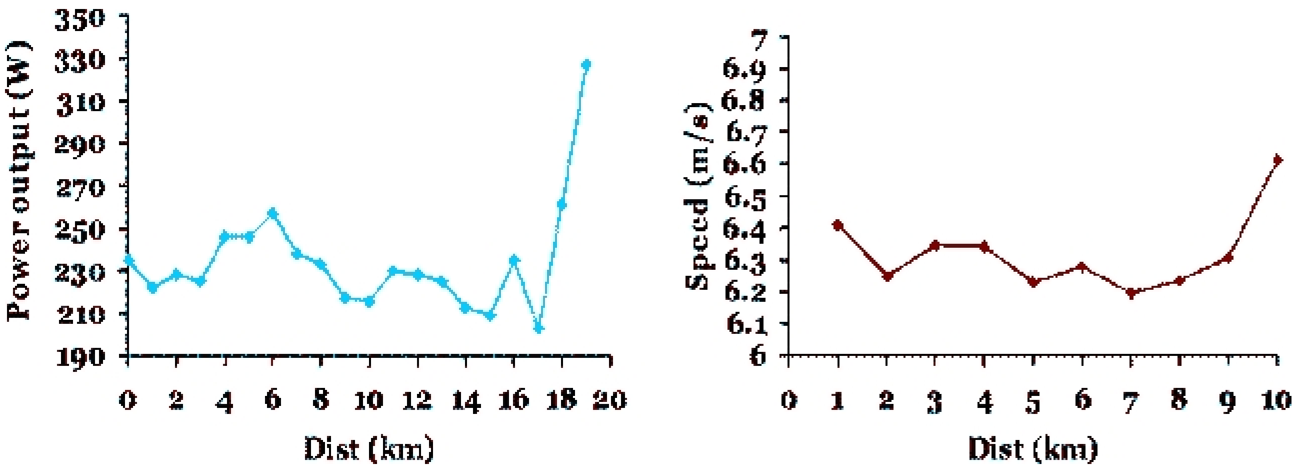

Figure 2: Left: Recorded power output of a club-level cyclist performing a fixed distance time trial on a bicycle. Right: The velocity plot of Haile Gebresellasie's world record 26:22 10000m on the track in 1998. Figure from: http://www.sportsscientists.com/2008/05/fatigue-and-exercise-part-i.html

4. Optimal Pacing with Uncertainty

Without compromising the simplicity Keller's model, we will revise the theory in a way which would lead very naturally to the fast-finish strategy which is often employed by seasoned competitors. We will begin by assuming that if the runner knows exactly how much energy is available for exercise before hand, E_0, and at exactly what rate it can be produced in the body, σ, the ideal strategy is the one of a constant cruising speed for almost the entirety of the race, as indicated by Keller in 1973 and most coaches today, even with a twenty-first century understanding of physiology.

However, on any given day the runner or coach does not know these physiological parameters with sufficient precision for an optimal performance. In the world of competitive athletics, one half-second in a race of one thousand seconds can make a significant difference. If a competitor were to apply Keller's model and assume that the value of σ is higher than it actually is, even slightly, the competitor would not be able to finish the race optimally and would slow down before the finish, a move that would lead to certain defeat. We can model this uncertainty in a variety of ways using the language of probability.

Let's consider a few examples and see what kind of behavior they predict.

5. A Probability Model

We retain all aspects of the Keller model, except we let E_0 be a normal random variable with a small variance. We then allow the runner to take a sample value of the random variable and apply Keller's strategy with that sample value of E_0 used to construct his strategy but the true E_0 value used to determine when he really runs out of energy. It follows that the time the runner will take to run a particular race will have some distribution with a minimum time which would correspond to the runner perfectly selecting the mean (true) value of E_0.

First we consider how the runner will perform, given different selections of E_0. We consider a discrete distribution of estimates, E_*, where $E_0 - .5 \times E_0 \leq E_* \leq E_0 + .5 \times E_0$ and the different values E_* are evenly spread over the given range. We solve equations (3.12-3.14) in (Keller 1974) with the different E_* values for E_0 to find our t_1 and λ under the different initial conditions. Then with those values we use equation (3.10) to find the cruising velocity, and we use equation (2.1) to find the cruising propulsive force per unit mass. Then we analyze the energy situation using equation (1.5) where f is constant (but not the same) in the both the acceleration and cruise regimes, and v is given by (Keller 1974, 3.2) in the acceleration regime, and constant in the cruise regime and given by τ/λ. We find the time when the runner is out of energy, and how far he has traveled in that time, summing the distance and the energy consumption from the first two regimes. From that point forward we require that he run at the minimum speed of 6.08 m/s which corresponds to a pace of energy balance in the Keller model. This is an approximation (but quite accurate) because the runner can take advantage of his initially higher speed and coast for a bit down to the minimum speed without consuming energy. These effects are minimal, however. The rate of energy in from breathing is equal to the rate out from running at that speed so the energy no longer decreases.

Given the complicated nature of Keller's equations 3.12-3.14, the best way to obtain the values of g for various E_* is numerically. A plot of the finish times of a 1000m race given 100 different values of E_* appears in Figure 3. The general form of the result is the same for all non-sprint distances. This function, which we will call g, where for each E_* there exists a T such that $g(E_*) = T$ is essentially piecewise linear as can be seen from Figure 3 and it has a minimum at E_0, as we would expect. Other distances produce similar results, but the linear pieces will have slightly different slopes.

We now aim to find an analytic expression for the distribution of finish times given that we assume the function g, which we have just described, is indeed piecewise linear and also given that the random variable from which the runner selects his estimate of E_0 is normally distributed with variance σ^2.

We proceed by seeking out the distribution of finish times given that the runner selects his E_* from a normal distribution with mean m and variance σ^2, m not necessarily equal to

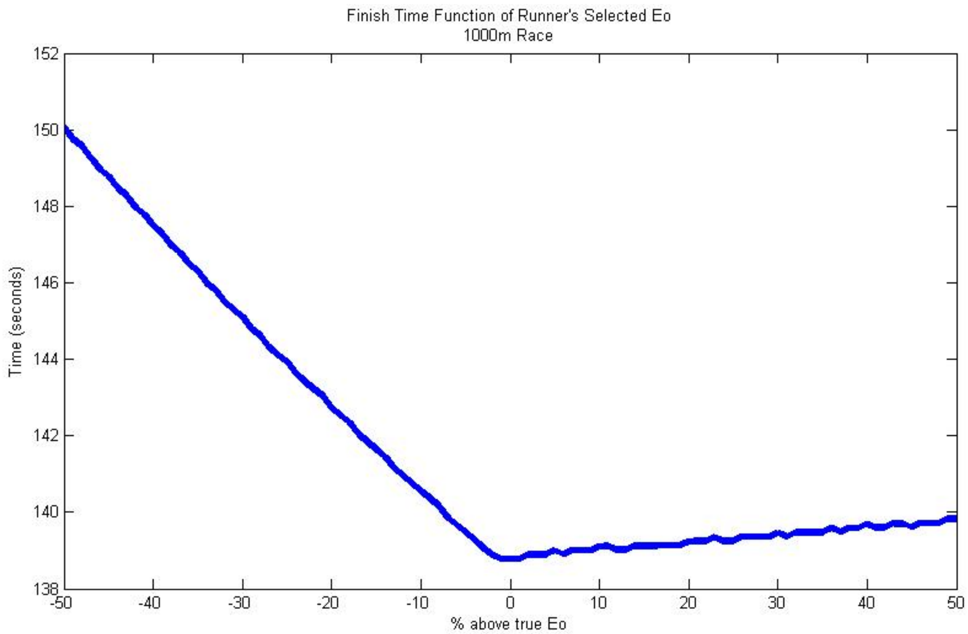

Figure 3: 1000m race finish times given different estimates of E_0 by the runner. In this model it is adventageous to start out a little fast since you can never learn at any stage whether you are running too slow.

E_0. Say the linear approximation of the left part of the curve in Figure 3 is given by the function $h(x) = c + dx$ and the right half of the curve is given by $f(x) = a + bx$ for constants $a, b, c, d, y \in \mathbb{R}$, where y is a particular finish time. Then we observe that the following relation holds:

$$P(T > y) = P\left(N(m, \sigma^2) > \frac{y-a}{b}\right) + P\left(N(m, \sigma^2) < \frac{y-c}{d}\right)$$

Figure 4 illustrates this situation. We then rewrite in terms of standard normals as follows and ultimately obtain an expression for the distribution of finish times:

$$P(T > y) = P\left(Z > \frac{((y-a)/b) - m}{\sigma}\right) + P\left(Z < \frac{((y-c)/d) - m}{\sigma}\right)$$

Again we rewrite using standard results about the normal distribution, and where Φ is the cumulative distribution function (cdf) of the standard normal random variable and ϕ is the probability density function (pdf) of the standard normal random variable:

$$F_T = P(T > y) = 1 - \Phi\left(\frac{((y-a)/b) - m}{\sigma}\right) + \Phi\left(\frac{((y-c)/d) - m}{\sigma}\right)$$

What we have obtained is F_T, the CDF of the random variable T, which is the finish time. Figure 5 illustrates this function for the 800m race distance.

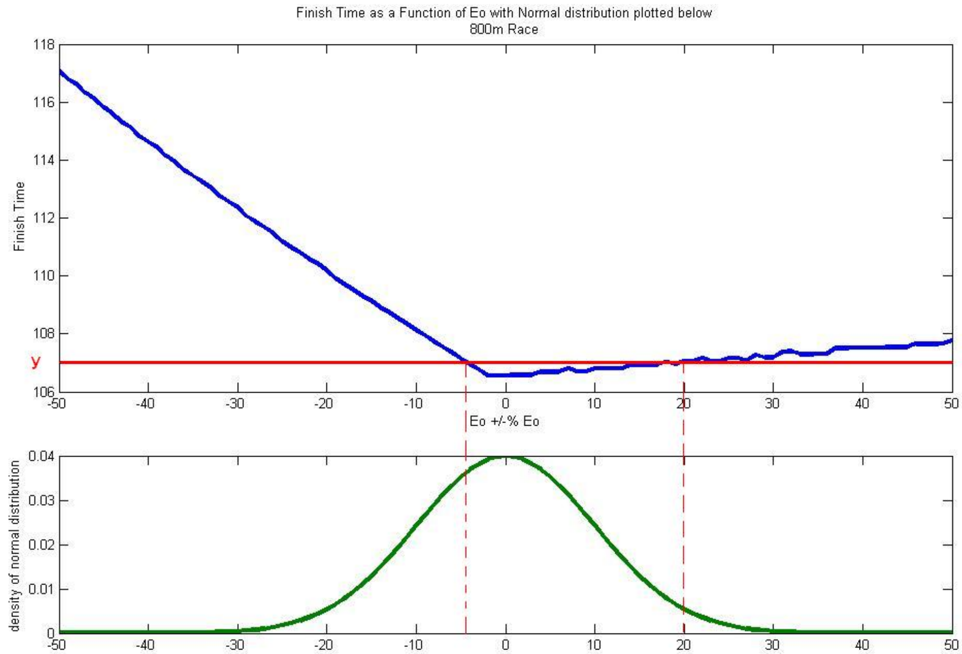

Figure 4: 800m race finish times given different estimates of E_0 by the runner. In this model it is apparent that it would be adventageous to take a risky (read high) estimate for E_0 since you can never learn at any stage whether you are running too slow. Note in this case, we assume the variance to be $\sigma^2 = E_0/10$

Now that we have obtained the cdf, the next logical goal is to find the pdf of T. We differentiate F_T to obtain f_T:

$$f_T = -\frac{1}{\sigma}\phi\left(\frac{((y-a)/b)-m}{\sigma}\right) + \frac{1}{\sigma}\phi\left(\frac{((y-c)/d)-m}{\sigma}\right)$$

f_T is plotted for an 800m race in figure 6. Now we have characterized the distribution. But, these results illuminate an immediate optimization problem. Suppose we could adjust the mean of our sampling distribution, to take advantage of the dramatically different results if the runner picks too slow or too fast. In this simplistic scenario, it is adventageous to bias your pick a little high to obtain the best result, since you can never accelerate if you end up starting out too slow. The question: how can we adjust the mean of our sampling distribution to minimize the runner's expected finish time?

We make the following definitions letting X be a normal random variable which corresponds to E_0 in the Keller model:

$$X \sim N(m, \sigma^2)$$

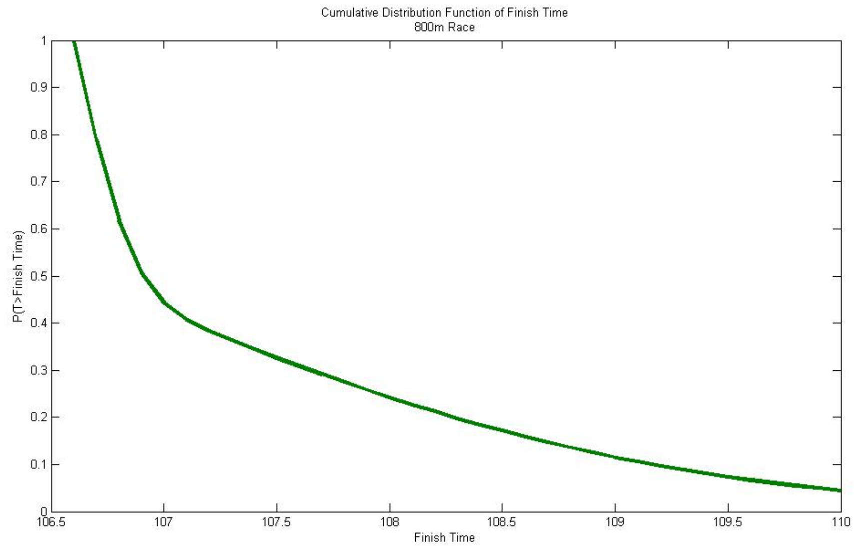

Figure 5: A plot of the cumulative distribution function of the random variable T which indicates the finish time. Note in this case, we assume the standard deviation to be $\sigma = E_0/10$ and $m = 0$

$$g(x) = f(x) = a + bx \quad x > 0$$

$$g(x) = h(x) = c + dx \quad x \leq 0$$

We now seek to find an analytic expression for the following expectation which would give us the expected finish time:

$$E(g(X)) = E(a + bX|X > 0) \times P(X > 0) + E(c + dX|X \leq 0) \times P(X \leq 0)$$

We remove the random variable X from the expectations and distribution functions on the right in the above equation and replace it with standard normal random variables after adjustment for the mean and variance of X:

$$E(a + bX|X > 0) = a + b\left[m + \sigma E\left(N(0,1)|N(0,1) > -m/\sigma\right)\right]$$

$$E(c + dX|X \leq 0) = c + d\left[m + \sigma E\left(N(0,1)|N(0,1) > -m/\sigma\right)\right]$$

$$P(X > 0) = P(Z > -m/\sigma) = 1 - \Phi(-m/\sigma) \quad \text{and} \quad P(X < 0) = \Phi(-m/\sigma)$$

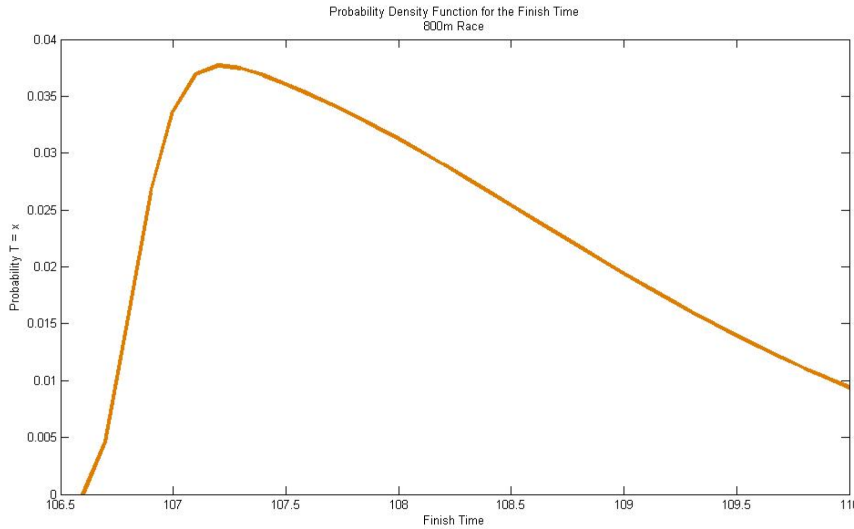

Figure 6: A plot of the probability density function, f_T. Note, in this case we assume the standard deviation to be $\sigma = E_0/10$ and $m = 0$

Now we make use the following general fact from the theory of probability ():

$$E(N(m,\sigma^2)|a < N(m,\sigma^2) < b) = m + \sigma \left(\frac{(\phi((a-m)/\sigma) - \phi((b-m)/\sigma))}{(\Phi((b-m)/\sigma) - \Phi((a-m)/\sigma))} \right)$$

We will show that the result is true for standard normal random variables without loss of generality. In the following, let Z be a standard normal random variable with cdf Φ and pdf ϕ as usual. Then:

$$E[Z|a < Z < b] = \frac{P(Z = z \quad \text{and} \quad a < z < b)}{P(a < Z < b)} = \frac{\int_a^b z\phi(z)dz}{\Phi(b) - \Phi(a)}$$

Note that $z\phi(z) = \frac{z}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = -\frac{d\phi(z)}{dz}$ so we substitute under the integrand in the expectation:

$$\frac{\int_a^b z\phi(z)dz}{\Phi(b) - \Phi(a)} = \frac{-\int_a^b \left(\frac{d\phi(z)}{dz}\right) dz}{\Phi(b) - \Phi(a)} = \frac{\phi(a) - \phi(b)}{\Phi(b) - \Phi(a)}$$

We use this result to make the following simplifications to the expectations:

$$E(a + bX|X > 0) = a + b\left[m + \sigma \frac{\phi(-m/\sigma^2)}{1 - \Phi(-m/\sigma^2)}\right]$$

$$E(c + dX|X \leq 0) = c + d\left[m + \sigma \frac{-\phi(-m/\sigma^2)}{\Phi(-m/\sigma^2)}\right]$$

As a result we have the following formula for the expectation, $E(g(X))$, we wished to find:

$$
\begin{aligned}
E(g(X)) &= (1-\Phi(-m/\sigma)) \times \left(a + b\left[m + \sigma\frac{\phi(-m/\sigma^2)}{1-\Phi(-m/\sigma^2)}\right]\right) \\
&+ \Phi(-m/\sigma) \times \left(c + d\left[m + \sigma\frac{-\phi(-m/\sigma^2)}{\Phi(-m/\sigma^2)}\right]\right)
\end{aligned}
\tag{5.1}
$$

We can use this formula to find the optimal adjustment to the mean of our distribution which the runner samples from to estimate E_0. As an example, consider the case of the 800m race. In this case, given a variance of $E_0/10$, the optimal mean of our sampling distribution is roughly $E_0 + E_0/10$ as illustrated by figure 7.

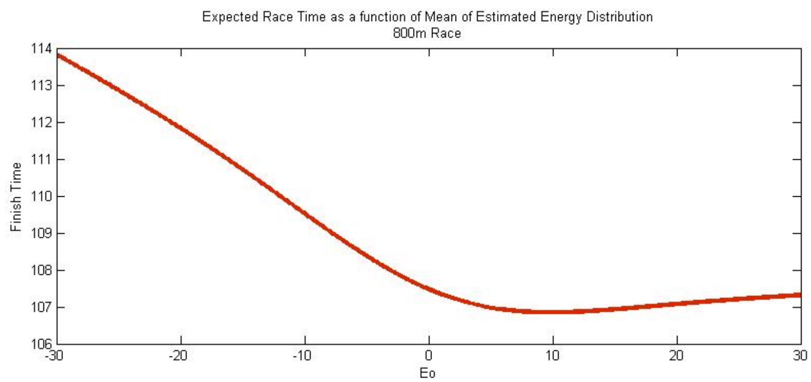

Figure 7: A plot of the expected finish time as a function of the selected mean of the sampling distribution of E_0. We assume the standard deviation to be $\sigma = E_0/10$

In general, after choosing a variance and choosing the race distance, one must determine the parameters a, b, c, d. This is non-trivial due to the complicated nature of Keller's equations. But, due to the approximately linear nature of the distribution of finish times over different values of E_0, one could obtain these values with only two additional evaluations of the three equations (one on each side of Keller's E_0) and then assuming a straight line through each of those points. In any case, after performing the linear fits to obtain a, b, c, d, we have shown that the expression for $E(g(X))$ is an analytic function of a single variable, m, which we can minimize over and find the optimal value of m.

This model leaves much to be desired though. We would like to allow the runner to learn his energy as he goes, in a more realistic way.

6. A Stochastic-Process Model

In Keller's model we primarily view all runners as the same. But, in reality this is not the case. Different runners have different values of the constants and depending on a particular runner's current health and fitness, the values of the constants may change slightly. The most significant constant in Keller's model which relates to distance running is σ. It is well known in exercise physiology that the runner's ability to consume oxygen is one of the best indicators of running performance although it is certainly not an infalible indicator. In fact, it has been shown that runners can improve their ability to consume oxygen from breathing with endurance training (see Jack Daniels, *The Running Formula* published in 2004 for a primer on basic running related physiology). With this in mind, another way to model uncertainty in this situation is to assume that σ is normally distributed about the constant value determined in Keller's 1973 paper.

Moreover, we could introduce an adaptive learning model where the runner learns about his true value of σ over the course of the race rather than sampling at the beginning and never again. Mathematically, we can express this in the following way: let the estimated value of σ at t, $\sigma_e(t)$, be a linear function of a Brownian motion:

$$\sigma_e(t) = \bar{\sigma} + KB_t$$

where $\bar{\sigma}$ is the constant Keller determined with his model, K is a constant which will adjust the variance of the model, and B_t is a standard one-dimensional Brownian motion. We will take the final value at time T of this stochastic process to be the true value of σ for that particular runner on that particular day.

Before moving forward, observe that for a race of known length T, the expected value of σ at any intermediate time $t < T$ is given by: $E_t[\sigma_T] = \sigma_t$ by the definition and basic properties of Brownian motion. The key facts are that it is a Gaussian process with independent increments. Furthermore, as t approaches T the variance of the random variable $E_t[\sigma_T]$ decreases to zero. As a result, the runner now has the option of adjusting his pace as his estimate for sigma improves since the variance will decline with the time. In short, the runner's best estimate for his true value of sigma is always the current estimate $\sigma_t = \bar{\sigma} + KB_t$.

To make an attempt at optimizing the runner's strategy, we formulate the problem in the language of stochastic control. We seek to maximize the expected value of D given T subject to the constraint that the expected value of the energy at time T is zero ($E(T) = 0$). We will let f_t be a stochastic process which is in our control. To simplify the situation, assume that $f_t = f(t, B_t)$, where f is a twice differentiable function on $[0, \infty) \times \mathbb{R}$. With these regularity conditions, we can properly define stochastic integrals.

Then we need to express the state of our system which can be defined by three variables, position, energy, and time:

$$dX_t = v_t dt$$

We still need an expression for v_t. Use the physical differential equation where $f(t, B_t)$ is now the stochastic process as described above:

$$\frac{dv}{dt} = -\frac{v}{\tau} + f(t, B_t)$$

We can solve this linear stochastic differential equation without difficulty using the method of integrating factors (in this case the factor is $e^{t/\tau}$). We obtain the following result after simplification:

$$v(t) = e^{-t/\tau} \int_0^t f(s, B_s) e^{s/\tau} ds$$

Then we have the following expression for X_t:

$$X_t = \int_0^T \int_0^t f(s, B_s) e^{s-t/\tau} dsdt$$

but we still need to have an expression for energy.

$$dE_t = \sigma_t(B_t) - v(t) \cdot f(t, B_t) = \sigma_t - f(t, B_t) e^{-t/\tau} \int_0^t f(s, B_s) e^{s/\tau} ds$$

Then we write our expression for our estimate for the current state of the energy:

$$E_t = E_0 + \sigma_t(B_t)t - \int_0^T \int_0^t f(t, B_t) f(s, B_s) e^{(s-t)/\tau} dsdt$$

We now have the following problem: Given T, we wish to maximize the expectation $E[X_T]$ such that the expectation $E[E_T] = 0$ by varying $f(t, B_t)$.

$$\max \left\{ E^f[X_T] \right\}$$

such that

$$E^f[E_T] = 0$$

Rewritten with the integrals:

$$\max_f E \left[\int_0^T \int_0^t f(s, B_s) e^{s-t/\tau} dsdt \right]$$

such that

$$E\left[[E_0 + T\cdot\sigma_T(B_T)] - \int_0^T\int_0^t f(t,B_t)f(s,B_s)e^{(s-t)/\tau}dsdt\right] = 0$$

Now we use a method of lagrange multipliers to create a stochastic process J_t which we wish to maximize (which is now unconstrained). See Oksendal Section 11.3 for a theorem and proof:

$$J_t = E\left[X_t + \lambda E_t\right]$$

We write in integral notation:

$$\begin{aligned} J_T &= E \int_0^T\int_0^t f(s,B_s)e^{(s-t)/\tau}dsdt \\ &+ \lambda\left[E_0 + T\cdot\sigma_T(B_T)\right] \\ &- \lambda\int_0^T\int_0^t f(t,B_t)f(s,B_s)e^{(s-t)/\tau}dsdt \end{aligned}$$

We recall that the true value of σ is a linear function of the final value of a Brownian motion B_T.

Then for simplicity, we first seek the optimal linear control $f(t,B_t)$, where $f(t,B_t) = a + bt + cB_t$ is linear. Then we seek:

$$\max J_T = \max E\int_0^T\int_0^t (1-\lambda f(t,B_t))f(s,B_s)e^{(s-t)/\tau}dsdt + \lambda\left[E_0 + T\cdot\sigma_T(B_T)\right]$$

$$= \max_{a,b,c} E\int_0^T\int_0^t (1-\lambda(a+bt+cB_t))(a+bs+cB_s)e^{(s-t)/\tau}dsdt + \lambda\left[E_0 + T\cdot\sigma_T(B_T)\right]$$

We now take expected values using the facts that $E[B_t] = 0$, $E[B_t\cdot B_s] = s$ for $0\le s\le t$, and $E[\sigma(B_t)] = \bar{\sigma}$ and then integrate. We obtain the following result from the computation:

$$\begin{aligned} \max J_T &= \max_{a,b,c}\big\{\tau\big[(-1/3)b^2\lambda T^3 + (1/2)T^2\cdot(\lambda\tau b^2 - 2ab\lambda + b - c^2\lambda) \\ &\quad +T(-\lambda a^2 + \tau\lambda ba + a - \tau b + \tau\lambda c^2) \\ &\quad +\tau^2 e^{-T/\tau}\cdot\left(\lambda b^2(T+\tau) + b(\lambda Ta - 1) + \lambda(a^2 - a + c^2)\right) \\ &\quad -\tau^2\left(\tau\lambda b^2 - b + \lambda(a^2 - a + c^2)\right)\big] + \lambda\left(E_0 + T\cdot\bar{\sigma}\right)\big\} \end{aligned}$$

This is now a just an optimization problem of the regular calculus which can be taken care of with the Lagrange multiplier method as in Keller's original papers for example, by finding a stationary point over all (a, b, c, λ). Arguments would also have to be made to prove that it is indeed the maximum value (and not just a stationary point over (a, b, c)).

There are several key points to observe about this analysis. First, the runner could potentially be operating with less than zero energy for some portion of the race in this model. The runner does not know his energy at any point and the constraint is only that the expected value of the energy is zero at the finish, not that energy is always positive. I propose to remedy this situation similarly to the way I remedied this problem in the probability model. Whenever the runner is at zero energy, he will be forced to drop to the minimum speed limit, which will negatively impact his pacing and make his results non-optimal in these cases. The effects of this remain to be studied. To fix this problem the model would have to be revised. It is possible that we might revise the model in such a way that the runner learns the true value of E as $E \to 0$ rather than the value of σ as $t \to 0$. This remains to be done in future work.

Furthermore, it is also important to note that an optimal linear control, although it can be analytically obtained, may not be very effective. In any case, it is an interesting analysis. As a reference for this section, see Bernt Oksendal's Book on Stochastic Differential Equations.

7. Conclusion

We have introduced some uncertainty into the Keller theory of running and attempted to model this behavior in a couple of ways. First, we introduced a basic probability model where the initial energy of the runner was a normal random variable, from which the runner sampled from to form his strategy. However, the true value of his energy is always a fixed constant as determined by Keller and, as a result, we could find the distribution of his finish times as a function of which value of E_0 the runner selects from the distribution. Given that the runner never has an opportunity to estimate his energy again, we found that it was advantageous for the runner to bias his pick a bit higher than the mean value, the extent depended upon the parameters.

To allow the runner to adapt and learn about his energy, we introduced a stochastic process model, where the runner learns about is his oxygen intake capacity, σ, over the course of the race. We then proved that an optimal linear control could be found through the methods of stochastic optimal control. It turns out that this model is not ideal because it allows for an unrealistic result, namely that the runner can operate below zero energy since the runner never knows how much energy he has until he finishes. One can tweak the model as I suggest to correct this problem by forcing the runner to be at the minimum speed limit during these times, but then the optimal control will no longer be optimal, since it does not take these adjustments into account.

In short, although a lot has been done, quite a bit more analysis needs to be done to achieve any meaningful results. There are several routes which could be taken to enhance the probability model, including adding additional sampling opportunities for the runner, finding the optimal location of these sampling points.

In the stochastic process model, there remain some serious difficulties. The model needs to be significantly revised to incorporate the fact that the runner can never be below zero energy. At the moment it is not clear how to do this while maintaining the uncertainty and adaptive learning features of the model, both of which I consider essential.

References

Davis, J. Mark & Bailey, Stephen P. (1997). Possible mechanisms of central nervous System fatigue during exercise. *Journal of Medicine & Science in Sports and Exercise*. 29(1):45-57.

Keller, J. B. (1973). A theory of competitive running. *Physics Today*. 26 (9):43-47.

Keller, J. B. (1974) Optimal velocity in a race. *American Mathematical. Monthly*, 81, 474-480.

Keller, J. B. Personal Communication.

Noakes, Tim. (2003). *The Lore of Running*. Champaign, IL: Human Kinetics.

Oksendal, Bernt. "Stochastic Differential Equations. An Introduction with Applications". Springer-Verlag. Sixth edition 2003.

Woodside, W. (1991) The optimal strategy for running a race. *Journal of Mathematical & Computer Modeling* 15:1–12